

ON SYMMETRIC CONTINUITY AND SYMMETRIC DIFFERENTIABILITY WITH RESPECT TO SETS

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Abstract

Full Text

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MATHEMATICS

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ON SYMMETRIC CONTINUITY AND SYMMETRIC DIFFERENTIABILITY WITH RESPECT TO SETS

(Presented by Academician P. S. Novikov on 13 IX 1968)

1. Let $f(x)$ be a real function defined on the interval $(0, 1)$; let k be an arbitrary fixed natural number;

$$\Delta^k f(x, h) = \sum_{i=0}^k (-1)^i \binom{k}{i} f[x + (k - 2i)h]$$

is the symmetric difference of order k at the point x ; and let Q be a set of points h on the line having zero as a limit point and situated to the right of zero.

We shall call $f(x)$ Q -symmetrically continuous of order k at the point $x \in (0, 1)$ if

$$\lim_{h \rightarrow 0, h \in Q} \Delta^k f(x, h) = 0.$$

We shall call $f(x)$ Q -symmetrically differentiable of order k at the point $x \in (0, 1)$ if there exists

$$D_Q^k f(x) = \lim_{h \rightarrow 0, h \in Q} \frac{\Delta^k f(x, h)}{(2h)^k}.$$

The upper and lower Q -symmetric derivative numbers are defined as follows:

$$\overline{D}_Q^k f(x) = \overline{\lim}_{h \rightarrow 0, h \in Q} \frac{\Delta^k f(x, h)}{(2h)^k}, \quad \underline{D}_Q^k f(x) = \underline{\lim}_{h \rightarrow 0, h \in Q} \frac{\Delta^k f(x, h)}{(2h)^k}.$$

If zero is an interior point for the set Q , then Q -symmetric continuity of order k coincides with the usual symmetric continuity of order k , while the definition

of the Q -symmetric derivative coincides with the definition of the Riemann derivative of order k . Recall that a function $f(x)$ has a Riemann derivative of order k at the point x if there exists

$$D^k f(x) = \lim_{h \rightarrow 0} \frac{\Delta^k f(x, h)}{(2h)^k}.$$

For $k = 1$ this definition coincides with the definition of the symmetric derivative; for $k = 2$, with the definition of the Schwarz derivative.

It is clear that ordinary continuity at a point x implies Q -symmetric continuity of any order k , and ordinary differentiability implies Q -symmetric differentiability of order 1 at this point x , whatever the set Q may be. The converse (for a fixed set Q) does not hold. However, as follows from the theorems given below, if one assumes that Q -symmetric continuity or Q -symmetric differentiability of order k of the function $f(x)$ holds at every point of a set E of positive measure, then, under certain conditions imposed on Q , it follows that $f(x)$ is ordinarily continuous or ordinarily differentiable almost everywhere on E .

We shall assume that for all x the set Q is one and the same. The idea of such a generalization is due to G. Kh. Sindalovskii. He introduced ⁽¹⁾ the concepts of continuity and differentiability with respect to congruent sets (the concepts of Q -continuity and Q -differentiability) and studied the connection of these concepts with the concepts of ordinary continuity and differentiability.

Let us introduce the following classes of sets Q having zero as their limit point.

To class (A) we assign those sets Q that have positive measure in every neighborhood of zero, i.e. such that $\text{mes } Q \cap (0, \delta) > 0$ for every $\delta > 0$.

To class (B) we assign those sets Q that have positive lower density at zero, i.e. those for which

$$\lim_{\delta \rightarrow 0} \frac{\text{mes } Q \cap (0, \delta)}{\delta} > 0.$$

To class (C) we assign those sets Q whose closure $\overline{Q}$ belongs to class (B).

Theorem 1. Let k be an arbitrary fixed natural number, $f(x)$ a function measurable on $(0, 1)$, for which

$$\lim_{\substack{h \rightarrow 0 \\ h \in Q}} \Delta^k f(x, h) = 0$$

at every point of a set E , $E \subset (0, 1)$, $\text{mes } E > 0$, and let the set Q belong to class (A). Then $f(x)$ is continuous almost everywhere on E .

For every set Q not belonging to class (A), there exists an everywhere on $(0, 1)$ symmetrically discontinuous* (and hence everywhere discontinuous in the ordinary sense) measurable function that will be Q -symmetrically continuous of order k almost everywhere on $(0, 1)$.

Theorem 2. Let k be an arbitrary fixed natural number, and let the set Q belong to class (B). If $f(x)$ is a function measurable on $(0, 1)$ for which

$$\lim_{\substack{h \rightarrow 0 \\ h \in Q}} |\Delta^k f(x, h)/(2h)^k| < +\infty$$

and $f'_{ac}(x)$ exists at every point $x \in E$, $E \subset (0, 1)$, $\text{mes } E > 0$, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E .

For every set Q not belonging to class (B), there exists a function measurable on $(0, 1)$ that has a Q -symmetric derivative of order k almost everywhere on $(0, 1)$ and nowhere has a symmetric (and hence ordinary) derivative.

Theorem 3. Let k be an arbitrary fixed natural number, and let the set Q belong to class (C). If $f(x)$ is a function continuous on $(0, 1)$ for which

$$\lim_{\substack{h \rightarrow 0 \\ h \in Q}} \left| \frac{\Delta^k f(x, h)}{(2h)^k} \right| < +\infty$$

and $f'_{ac}(x)$ exists at every point $x \in E$, $E \subset (0, 1)$, $\text{mes } E > 0$, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E .

For every set Q not belonging to class (C), one can construct a function continuous on $(0, 1)$ having a finite Q -symmetric derivative of order k on some set E , $E \subset (0, 1)$, and having no ordinary derivative on it. The measure of the set E may be arbitrarily close to one.

For $k = 1$ the following assertions are valid:

Theorem 4. Let $f(x)$ be a function measurable on $(0, 1)$; Q a set belonging to class (B). Then $\overline{D}_Q f(x)$ almost everywhere on $(0, 1)$ is equal to $\overline{D}f(x)$.

For every set Q not belonging to class (B), there exists a function $\varphi(x)$, measurable on $(0, 1)$ and nowhere symmetrically differentiable, for which almost everywhere on $(0, 1)$ the relation

$$\overline{D}_Q \varphi(x) < \overline{D} \varphi(x) = +\infty$$

holds (here $\overline{D}f(x)$ is the upper symmetric derivative number of the function $f(x)$).

Theorem 5. Let $f(x)$ be a function continuous on $(0, 1)$; Q a set—

* A function $f(x)$ is called everywhere symmetrically discontinuous on $(0, 1)$ if for every $x \in (0, 1)$ $\Delta^1 f(x, h)$ does not tend to zero as $h \rightarrow 0$.

a set belonging to class (C). Then almost everywhere on $(0, 1)$

$$\overline{D}_Q f(x) = \overline{D}f(x).$$

For every set Q not belonging to class (C), there exists a continuous function $\varphi(x)$, nowhere differentiable on $(0, 1)$, for which almost everywhere the relation

$$\overline{D}_Q \varphi(x) < \overline{D} \varphi(x) = +\infty$$

holds.

Analogous assertions hold for the lower symmetric derived numbers.

Theorem 6. Let $f(x)$ be a measurable function on $(0, 1)$; let Q be a set belonging to class (B). If at each point $x \in E \subset (0, 1)$ at least one of the inequalities

$$\overline{D}_Q f(x) < +\infty$$

or

$$\underline{D}_Q f(x) > -\infty$$

holds, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E .

For every set Q not belonging to class (B), there exists a measurable symmetric function $\varphi(x)$, nowhere symmetrically differentiable on $(0, 1)$, for which almost everywhere on $(0, 1)$ the inequality

$$\overline{D}_Q \varphi(x) < +\infty$$

holds.

Theorem 7. Let $f(x)$ be a continuous function on $(0, 1)$; let Q be a set belonging to class (C). If at each point $x \in E \subset (0, 1)$ at least one of the inequalities

$$\overline{D}_Q f(x) < +\infty$$

or

$$\underline{D}_Q f(x) > -\infty$$

holds, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E .

For every set Q not belonging to class (C), there exists a function $\varphi(x)$ continuous on $(0, 1)$, having no ordinary derivative, but satisfying the inequality

$$\overline{D}_Q \varphi(x) < +\infty$$

almost everywhere on $(0, 1)$.

2. The proof of Theorem 2 is based on the following assertions:

Lemma 1. Let k be a fixed natural number; let $f(x)$ be a function measurable in the sense of Lebesgue on $(0, 1)$; let Q be a set belonging to class (B); and let F be a Borel-measurable set such that: a) $F \subset (0, 1)$, $\text{mes } F = 1$; b) $f(x)$ is Borel-measurable on F ; c) for every $x \in CF$ ($CF = (0, 1) \setminus F$) there exists a sequence of numbers $\{c_i\}$, $c_i \in F$, such that $c_i \rightarrow x$ and $f(c_i) \rightarrow f(x)$ as $i \rightarrow \infty$. If

$$\overline{D}_Q^k f(x) < +\infty$$

for $x \in E$, $E \subset (0, 1)$, $\text{mes } E > 0$, then there exist a perfect set P , $P \subset E \cap F$, of measure arbitrarily close to the measure of E , and numbers $\eta_0 > 0$, $l_0 > 0$, such that

$$\Delta^k f(x, h)/(2h)^k < \eta_0$$

for $x \in P$, $h \in Q$, $h < l_0$, such that $x \pm rh \in F$ ($1 \leq r \leq k$).

Lemma 1 is proved by the methods developed by G. Kh. Sindalovskii ^(1,2). The first part of Theorem 3 follows from the first part of Theorem 2. Theorem 4 is based on the following assertion:

Lemma 2. Let $f(x)$ be a measurable function on $(0, 1)$; let Q be a set belonging to class (B). If

$$\overline{D}_Q f(x) < +\infty$$

on some set E of positive measure, then almost everywhere on E $f(x)$ is asymptotically differentiable. An analogous assertion holds if

$$\underline{D}_Q f(x) > -\infty.$$

Theorems 5 and 6 follow from Theorem 4; Theorem 7 follows from Theorem 6.

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CITED LITERATURE

1. G. Kh. Sindalovskii, *Izv. AN SSSR, Ser. Matem.*, **26**, 125 (1962).
2. G. Kh. Sindalovskii, *Izv. AN SSSR, Ser. Matem.*, **24**, 707 (1960).

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