

EXPONENTIAL DECAY OF THE SOLUTION OF THE WAVE EQUATION IN THE EXTERIOR OF AN ALMOST STAR-SHAPED DOMAIN

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Abstract

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MATHEMATICS

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EXPONENTIAL DECAY OF THE SOLUTION OF THE WAVE EQUATION IN THE EXTERIOR OF AN ALMOST STAR-SHAPED DOMAIN

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1. In this paper we study the exponential decay as $t \rightarrow \infty$ of the solution of the mixed problem

$$u_{tt} - \Delta u = 0, \quad x \in C\bar{B}, \quad t > 0, \quad (1)$$

$$u|_{\partial B} = 0, \quad t > 0, \quad (2)$$

$$u|_{t=0} = f_0, \quad u_t|_{t=0} = f_1, \quad x \in CB \quad (f_0|_{\partial B} = 0), \quad (3)$$

where B is an open bounded domain of odd-dimensional Euclidean space R^l ($l > 1$), $x = (x_1, \dots, x_l)$ are the spatial variables; CB , the complement of $\bar{B}$, is connected; ∂B , the boundary of B , is of class C^2 ; B contains the origin. Let R be such that the ball $S(R) = \{x : |x| < R\}$ contains $\bar{B}$. Introduce the energy norms

$$\|u\|_{E,t}^2 = \int_{CB} \left(|u_t|^2 + \sum_{i=1}^l |u_{x_i}|^2 \right) dx,$$

$$\|u\|_{E,t,R}^2 = \int_{CB \cap S(R)} \left(|u_t|^2 + \sum_{i=1}^l |u_{x_i}|^2 \right) dx$$

and the norm on the set of initial data $f = \{f_0, f_1\}$

$$\|f\|_E^2 = \int_{CB} \left(|f_1|^2 + \sum_{i=1}^l |f_{0x_i}|^2 \right) dx.$$

Define the spaces of initial data H and $H(R)$ as the closures in the energy norm $\|\cdot\|_E$ of the sets $C_0^\infty(CB)$ and $C_0^\infty(CB \cap S(R))$, respectively.

It is known that if u is a solution of (1)–(3), $f \in H$, then $\|u\|_{E,t}$ is finite, does not depend on t , and is equal to $\|f\|_E$. It is also known ⁽⁴⁾ that in this case

$$\lim_{t \rightarrow \infty} \|u\|_{E,t,R} = 0. \quad (4)$$

However, uniformity of this convergence to zero over the set of initial data $f \in H(R)$ is far from always attained, i.e., the validity, for u the solution of (1)–(3), of the inequality

$$\|u\|_{E,t,R} \leq \varepsilon_R(t) \|f\|_E, \quad f \in H(R), \quad \varepsilon_R(t) \rightarrow 0, \quad t \rightarrow \infty, \quad (5)$$

where $\varepsilon_R(t)$ does not depend on f .

Examples of domains for which uniformity does not hold can be obtained from considerations of geometric optics. These are domains such that, for a given $S(R) \supset B$ and for arbitrarily large T , there exist a point $x \in S(R) \cap CB$ and a direction θ for which the ray emitted from x in the direction θ and reflected from ∂B according to the laws of geometric optics will remain in $S(R)$ for a time greater than T .

Although there is a conjecture that in all other cases uniformity holds, a positive result for the first boundary-value problem has been proved only for star-shaped B ⁽¹⁾.

It is also known ⁽²⁾ that for any domain estimate (5) entails

$$\|u\|_{E,t,R} \leq c_R \exp(-a_R t) \cdot \|f\|_E, \quad a_R > 0; \quad (6)$$

a_R, c_R do not depend on f, t ; u is the solution of (1)–(3) with initial data $f \in H(R)$.

In the present paper we introduce a broader class of domains than the star-shaped ones, and establish for them result (6).

2. Definition. A bounded open domain B with boundary of class C^1 is called **almost star-shaped** if there exist a D -bounded open neighborhood $\overline{B}$, a single-valued function $\varphi \in C^2(\overline{D} \cap CB)$, and a constant c_0 such that:

- (I) $\varphi(x) < c_0$, $x \in D \cap CB$, $\varphi(x) = c_0$, $x \in \partial D$.
- (II) $|\text{grad } \varphi(x)| \geq \text{const} > 0$, $x \in \overline{D} \cap CB$.
- (III) The level surfaces $\varphi(x) = c$ are strictly convex; the radius of curvature in all directions at all points of $CB \cap \overline{D}$ is uniformly bounded above.
- (IV) At the points of intersection of the level surfaces with ∂B , their outer normals form with the outer normal to ∂B no more than a right angle.

Remark 1. If the level surfaces are spheres with a common center, then B is star-shaped, and conversely. Almost star-shaped domains are a natural generalization of star-shaped ones.

Remark 2. In the definition we do not require the connectedness of CB , since it follows from (I)–(IV).

3. **Theorem 1.** *If l is odd and greater than 1, and B is almost star-shaped with boundary of class C^2 , then for any R there exist constants $a_R > 0, c_R$ such that for the solution of problem (1)–(3) with $f \in H(R)$ inequality (6) holds.*

Theorem 1 follows from Lemma 1 and from the result that estimate (5) entails estimate (6). Using the methods of semigroup theory developed in ⁽⁴⁾, one can establish that a stronger statement holds.

Theorem 2. *Under the assumptions of Theorem 1 there exist constants $c, \alpha > 0$ such that for all solutions of problem (1)–(3) with $f \in H(R_1)$*

$$\|u\|_{E,t,R_2} \leq c \exp(-\alpha t + \alpha R_1 + \alpha R_2) \|f\|_E \quad (7)$$

for any t, R_1, R_2 .

4. **Lemma 1.** *Under the assumptions of Theorem 1, inequality (5) holds.*

Lemma 2. *If B is almost star-shaped, then for any R there exists a constant c such that*

$$\|u\|_{E,t,R}^2 \leq \frac{c}{t} \|f\|_E^2 + c \sup_{0 \leq \tau \leq t} \|u(\tau)\|_{L_2(CB)}^2, \quad (8)$$

if u is the solution of problem (1)–(3), $f \in H(R)$, $t \geq 1$.

Lemma 3 (geometric). *If B is almost star-shaped, then there exist single-valued functions $\psi \in C^2(CB)$ and $\chi \in C^2(CB)$ such that:*

$$(I) \quad \frac{1}{2} + \frac{1}{2} \Delta \psi(x) - \sum_{i,j=1}^l \psi_{x_i x_j} \xi_i \xi_j \leq \chi(x) \leq -\frac{1}{2} + \Delta \psi(x), \quad x \in CB,$$

$$\xi \in R^l, \quad |\xi| = 1.$$

$$(II) \quad \text{grad } \psi(x) \cdot n(x) \geq 0, \quad x \in \partial B, \quad n(x) \text{ is the outer normal to } \partial B$$

at the point x .

(III) There exists a constant $\bar{c}$ such that

$$|\text{grad } \psi(x)| \leq |x| - R + \bar{c}, \quad |\chi(x)| \leq \bar{c}, \quad |\Delta \chi(x)| \leq \bar{c}, \quad x \in CB;$$

R is fixed.

We do not give the proof of Lemma 3 here. Instead of requirements (I)–(IV) in the definition of almost star-shapedness, one could have used assertions (I)–(III) of Lemma 3, but they seem unnatural to us from the geometric point of view.

The derivation of Lemma 2 from Lemma 3 can be carried out by multiplying equation (1) by $(2\bar{c} + t)\bar{u}_t + \sum_{i=1}^l \psi_{x_i} \bar{u}_{x_i} + \chi \bar{u}$, where $\psi, \chi, \bar{c}$ are obtained in Lemma 3. The resulting product is integrated over $x \in CB$, $0 \leq t \leq t_1$; (2), (3), the properties (I)–(III) of $\psi, \chi, \bar{c}$, and also the fact that f belongs to $H(R)$, are used.

5. Proof of Lemma 1. Define the operator $S : H(R) \rightarrow H$ as follows: $h = Sf$, $h = \{h_0, h_1\}$, where $h_1 = f_0$, h_0 is the solution of $\Delta h_0 = f_1$, $x \in CB$, $h_0|_{\partial B} = 0$, $h \in H$ (in general, the membership $h \in H$ entails $h_0|_{\partial B} = 0$); $h \in H$ is defined uniquely for any $f \in H(R)$.

From the chain of inequalities

$$\sum_{i=1}^l \|h_0, x_i\|_{L_2(CB)}^2 = -(h_0, \Delta h_0)_{CB} = -(h_0, f_1)_{CB} \leq \|h_0\|_{L_2(S(R) \cap CB)} \|f_1\|_{L_2(CB)}$$

and from the definitions of H , $H(R)$, S , it follows that S is bounded and completely continuous. Let v be the solution of (1)–(3) with initial data Sf ; then $u = v_t$, $\|u(\tau)\|_{L_2(CB)} = \|v_\tau(\tau)\|_{L_2(CB)} \leq \|Sf\|_E$, whence, and from equality (8), it follows that

$$\|u\|_{E,t,R}^2 \leq \frac{c}{t} \|f\|_E^2 + c \|Sf\|_E^2 \quad (t \geq 1). \quad (9)$$

Now let $\varepsilon > 0$ be given. Then $H(R) = H^{(1)} \oplus H^{(2)}$, where, for every $f^{(1)} \in H^{(1)}$, the inequality $\|Sf\|_E^2 \leq \frac{\varepsilon}{4c} \|f\|_E^2$ holds, and $H^{(2)}$ is finite-dimensional. Then from (4) and the finite-dimensionality of $H^{(2)}$ there follows the existence of T' such that, for $t \geq T'$,

$$\|u^{(2)}\|_{E,t,R}^2 \leq \frac{\varepsilon}{2} \|f^{(2)}\|_E^2$$

for all $f^{(2)} \in H^{(2)}$ and $u^{(2)}$ the solution of problem (1)–(3) with initial data $f^{(2)}$. Let $f \in H(R)$, and let u be the solution of (1)–(3) with initial data f , $f = f^{(1)} + f^{(2)}$, $f^{(k)} \in H^{(k)}$; then $u = u^{(1)} + u^{(2)}$, where $u^{(k)}$ is the solution of (1)–(3) with initial data $f^{(k)}$,

$$\|u\|_{E,t,R}^2 \leq 2(\|u^{(1)}\|_{E,t,R}^2 + \|u^{(2)}\|_{E,t,R}^2) \leq \varepsilon(\|f^{(1)}\|_E^2 + \|f^{(2)}\|_E^2) = \varepsilon \|f\|_E^2$$

for $t \geq \max(1, T', 4c/\varepsilon)$. (We have used the definition of $H^{(k)}$ and the inequalities for $\|u^{(k)}\|_{E,t,R}$.) The lemma is proved.

6. There are examples showing that the class of almost star-shaped domains is very precise for the problem under consideration. Thus, domains of

the type of an open horseshoe (with diverging sides) are almost star-shaped (but, generally speaking, not star-shaped). The limiting domain—the horseshoe with parallel sides—is not almost star-shaped, and the main result is false for it. Star-shaped domains do not allow examples of this kind.

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