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Abstract

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MATHEMATICS

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ON A CONDITION FOR REGULARITY OF A GAUSSIAN STATIONARY PROCESS

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1. Let $\xi(t)$ be a stationary, in the narrow sense, process with discrete or continuous time. Denote by $\mathfrak{M}_a^b$ the σ -algebra of events generated by the course of the process on the interval $[a, b]$, $-\infty \leq a < b \leq \infty$. A. N. Kolmogorov proposed the following condition of weak dependence of events generated by the past and the future of the process (see ⁽¹⁾):

$$\beta(\tau) = E \left\{ \sup_{A \in \mathfrak{M}_{-\infty}^0} |P\{A \mid \mathfrak{M}_{\tau}^{\infty}\} - P\{A\}| \right\} \xrightarrow{\tau \rightarrow \infty} 0. \quad (1)$$

Denote by $P_{0,\tau}\{\cdot\}$ the measure induced by the process $\xi(t)$ on the σ -algebra $\mathfrak{M}_{-\infty}^0 \cup \mathfrak{M}_{\tau}^{\infty}$, and by $P_{1,\tau}\{\cdot\}$ the measure on the same σ -algebra defined, for $A \in \mathfrak{M}_{-\infty}^0$, $B \in \mathfrak{M}_{\tau}^{\infty}$, by the equality

$$P_{1,\tau}\{AB\} = P_{0,\tau}\{A\}P_{0,\tau}\{B\}.$$

V. A. Volkonskii and Yu. A. Rozanov ⁽²⁾ found that

$$\beta(\tau) = \frac{1}{2} \text{Var}[P_{0,\tau} - P_{1,\tau}]. \quad (2)$$

Starting from (2), they obtained conditions, expressed in terms of the spectral density (s.d.), sufficient for a stationary Gaussian process with discrete time to satisfy condition (1). In the present note we give a scheme of proof of Theorem 1 formulated below, which makes it possible to obtain a complete description of stationary Gaussian processes with discrete time satisfying condition (1).

Theorem 1. *In order that a stationary Gaussian process with discrete time satisfy condition (1), it is necessary and sufficient that it have an s.d. $f(\lambda)$ representable in the form*

$$f(\lambda) = |P(e^{i\lambda})|^2 f_1(\lambda), \quad (3)$$

where:

- 1) $P(z)$ is a polynomial with roots on the circle $|z| = 1$;
- 2)

$$\ln f_1(\lambda) \sim \sum_{-\infty}^{\infty} f_k e^{ik\lambda},$$

where

$$\sum_{-\infty}^{\infty} |k| |f_k|^2 < \infty. \quad (4)$$

The proof of this theorem is contained in §§ 2, 3. There also some results are given for processes with continuous time. In § 4 results are given on the rate of decrease of $\beta(\tau)$.

2. We note that if a stationary, in the narrow sense, process satisfies condition (1), then it is regular in the sense of A. N. Kolmogorov (see ⁽¹⁾) and, consequently, has an s.d. $f(\lambda)$ satisfying, in the case of discrete time,

$$\int_{-\pi}^{\pi} \ln f(\lambda) d\lambda > -\infty. \quad (5)$$

It follows from condition (5) (see (3)) that there exists an outer function $g(z) \in H^2$ inside the unit disk such that $|g(e^{i\lambda})|^2 = f(\lambda)$. Put

$$G(\lambda) = \frac{\overline{g(e^{i\lambda})}}{g(e^{i\lambda})} \sim \sum_{-\infty}^{\infty} g_k e^{ik\lambda}. \quad (6)$$

Theorem 2. *In order that a stationary Gaussian process with discrete time satisfy condition (1), it is necessary and sufficient that it have a spectral density $f(\lambda)$ satisfying condition (5), and that the function $G(\lambda)$, defined in (6), be such that*

$$\sum_{-\infty}^0 |k| |g_k|^2 < \infty. \quad (7)$$

In the case of a process with continuous time, regularity implies the existence for the process of a spectral density $f(\lambda)$ satisfying the condition

$$\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1 + \lambda^2} d\lambda < \infty. \quad (5')$$

It follows from condition (5') (see (3)) that there exists an outer function $g(z) \in H^2$ in the right half-plane such that $|g(i\lambda)|^2 = f(\lambda)$. Put

$$G(\lambda) = \frac{\overline{g(i\lambda)}}{g(i\lambda)} \frac{1}{1 - i\lambda} \sim \int_{-\infty}^{\infty} \hat{G}(s) e^{is\lambda} ds. \quad (6')$$

Theorem 2'. *In order that a Gaussian stationary process (t) with continuous time satisfy condition (1), it is necessary and sufficient that it have a spectral density $f(\lambda)$ satisfying condition (5'), and that the function $G(\lambda)$, defined in (6'), be such that*

$$\int_{-\infty}^{\infty} |\hat{G}(s)|^2 |s| ds < \infty. \quad (7)$$

We give the scheme of the proof of Theorem 2.

Necessity. From condition (1) it follows that the process $\xi(t)$ has a spectral density $f(\lambda)$ satisfying condition (5). Denote by $H_a^b(i)$, $i = 0, 1$, $-a, b \geq 0$, the Hilbert space of random variables $\eta(\omega)$ of the form

$$\eta = \sum_{t \in [a, b]} c(t) \xi(t)$$

with scalar product

$$(\eta_1, \eta_2) = \int_{\Omega} \eta_1(\omega) \overline{\eta_2(\omega)} P_{i, \theta} \{d\omega\}.$$

Consider the linear operator B_n from $H_0^n(0)$ into $H_0^n(1)$ such that $(x, y)_0 = (B_{nx}, y)_1$, $x, y \in H_0^n(i)$. We calculate $\|B_n - E\|$ —the Hilbert-Schmidt norm of the operator $B_n - E$. If T is a linear isometry of the space $H_0^n(0)$ into $L_2(-\pi, \pi)$ such that $T\xi(t) = e^{it\lambda} g(\lambda)$, then, by Beurling's theorem ((3), p. 101), we have $TH_{-\infty}^n(i) = e^{in\lambda} H^2$. It is easy to see that $TH_0^\infty = \frac{g}{\bar{g}} H^2$. Hence

$$\|B_n - E\|^2 = \sum_k |g_{k+n}|^2 k. \quad (8)$$

From condition (1) follows the mutual absolute continuity of the Gaussian measures $P_{0,n}$, $P_{1,n}$ for sufficiently large n . Hence, using one

Feldman's result (see (4)), we infer that $\|B_n - E\| < \infty$. From the last inequality and relation (8), (7) follows.

Sufficiency. Relation (8) implies the mutual absolute continuity of the measures mentioned above, if $H_0^\infty(0) \cap H_{-\infty}^n = \{0\}$. Therefore, from relations (7), (8), and Theorem 4.1 of (2), relation (1) follows at once.

3. We finish the proof of **Theorem 1**.

Sufficiency. It is verified directly that (7) follows from (3).

Necessity. From condition (1) it follows that

$$\alpha(\tau) = \sup_{A \in \mathfrak{M}_{-\infty}^0, B \in \mathfrak{M}_{\tau}^{\infty}} |P\{AB\} - P\{A\}P\{B\}| \xrightarrow{\tau \rightarrow \infty} 0. \quad (9)$$

Denote by $L_2(f)$ the Hilbert space of functions square-integrable with weight f , with scalar product

$$(\varphi, \psi)_f = \int_{-\pi}^{\pi} \varphi(\lambda) \overline{\psi(\lambda)} f(\lambda) d\lambda.$$

By a theorem of A. N. Kolmogorov and Yu. A. Rozanov ⁽⁵⁾, condition (9) is equivalent to the condition

$$\rho(\tau; f) = \sup_{\varphi, \psi} |(e^{i\tau\lambda}\varphi, \overline{\psi})_f| \xrightarrow{\tau \rightarrow \infty} 0,$$

where the supremum is taken over all polynomials $\varphi, \psi \in H^2$ with norms $\|\varphi\|_f = \|\psi\|_f = 1$. Hence, on the basis of the results of papers ^(6,7), it follows that the function $f(\lambda)$ can be written in the form $|P(e^{i\lambda})|^2 f_1(\lambda)$, where $P(z)$ is a polynomial with roots on the circle $|z| = 1$, $\rho(1; 1/f_1) = \rho < 1$. It is not difficult to see that $\overline{P}/P = ae^{-ik\lambda}$, $|a| = 1$, k is the degree of P , so that together with the function $g(z)$ satisfying condition (7), the outer function $g_1(z)$, where $|g_1(e^{i\lambda})|^2 = f_1(\lambda)$, satisfies condition (7). The latter condition can be written in the form

$$\sum_A \inf \left\| \frac{\overline{g_1}}{g_1} - e^{-in\lambda} A \right\|_2^2 < \infty, \quad (10)$$

where $\|\cdot\|_2$ is the norm in $L_2(-\pi, \pi)$, and the infimum is taken over all $A \in H^2$. Choose a sequence of polynomials $A_n \in H^2$ for which the series (10) converges. Put $e^{-in\lambda} g_1 A_n = Q_n(e^{-in\lambda}) + B_n$, where Q_n is a polynomial of degree n , and all $B_n \in H^2$. Then

$$\left\| \overline{g_1}/g_1 - e^{-in\lambda} A_n \right\|^2 = \|(g_1 - Q_n) - B_n\|_{1/f_1}^2 \geq (1 - \rho) \|g - \overline{Q_n}\|_{1/f_1}^2.$$

Consequently, the series

$$\sum_{P_n} \inf \|g_1 - \overline{P_n}\|_{1/f_1}^2, \quad (11)$$

converges, where the infimum is taken over all polynomials $P_n(e^{i\lambda})$ of degree not exceeding n .

Introduce the polynomials $\varphi_\nu(z)$, orthonormal with weight $1/f_1$. From the properties of such polynomials ⁽⁸⁾ one can infer that convergence of the series (11) is equivalent to the inequality

$$\sum_1^\infty \nu |\varphi_\nu(0)|^2 < \infty.$$

Let $\varphi_\nu^*(z) = z^\nu \overline{\varphi_\nu(1/z)}$. The polynomials $\varphi_{\nu,n}(z)$, orthonormal with weight $|\varphi_n^*(e^{i\lambda})|^{-2}$, have the form $\varphi_{\nu,n}(z) = \varphi_\nu(z)$, $\nu \leq n$; $\varphi_{\nu,n}(z) = z^{\nu-n} \varphi_n(z)$, $\nu \geq n$. Consequently,

$$\sum_{\nu=1}^\infty \nu |\varphi_{\nu,n}(0)|^2 \leq \sum_{\nu=1}^\infty \nu |\varphi_\nu(0)|^2. \quad (12)$$

Denote by $D_{\nu,n}$ the ν -th Toeplitz determinant constructed with respect to the weight $|\varphi_n^*|^{-2}$. Let

$$G(h) = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln h(\lambda) d\lambda \right\}.$$

By a theorem of G. Szegő ⁽⁸⁾, p. 101),

$$\lim_{\nu \rightarrow \infty} \frac{D_{\nu,n}}{(G(|\varphi_n^*|^{-2}))^{\nu+1}} = \exp \left\{ \frac{1}{\pi} \iint_{|z| \leq 1} \left| \frac{\varphi_n^{*'}(z)}{\varphi_n^*(z)} \right|^2 d\sigma \right\}.$$

From this and from (12) it follows that all the integrals on the right-hand side of (13) are uniformly bounded. As $n \rightarrow \infty$, $\varphi_n^*(z) \rightarrow g_1(z)$ uniformly in every disk $|z| \leq r < 1$. Therefore

$$\sum_{-\infty}^\infty |k| |f_k|^2 = \frac{2}{\pi} \iint_{|z| \leq 1} \left| \frac{g_1'(z)}{g_1(z)} \right|^2 d\sigma = \lim_{n \rightarrow \infty} \frac{2}{\pi} \iint_{|z| \leq 1} \left| \frac{\varphi_n^{*'}(z)}{\varphi_n^*(z)} \right|^2 d\sigma < \infty.$$

4. In the case of continuous time, one can give conditions sufficient for condition (1) to hold.

Theorem 3. Let the spectral density $f(\lambda)$ of a stationary Gaussian process with continuous time be represented in the form $|B(i\lambda)|^2 f_1(\lambda)$, where $B(z)$ is an entire function of finite degree, and the function $f_1(\lambda)$ is such that: 1) $0 < m \leq f_1(\lambda) \leq M$; 2) the function $\hat{f}(\lambda)$ —the Fourier transform of $f_1(\lambda)/(1 + \lambda^2)$ —satisfies the inequality

$$\int_{-\infty}^{\infty} |\lambda| |\hat{f}(\lambda)|^2 d\lambda < \infty.$$

Then the process $\xi(t)$ satisfies condition (1).

In the case of a process with discrete time, one can, just as was done in paper ⁽⁹⁾ (see Theorems 3, 4, 5), completely describe the class of those spectral densities for which $\beta(n)$ decreases sufficiently rapidly. We give one such theorem.

Theorem 4. In order that $\beta(n) = O(n^{1/2-r-\alpha})$, where r is an integer, $r + \alpha > 1/2$, $0 < \alpha < 1$, it is necessary and sufficient that the spectral density $f(\lambda)$ be representable in the form $|P(e^{i\lambda})|^2 f_1(\lambda)$, where $P(e^{i\lambda})$ is a trigonometric polynomial, and the function $f_1(\lambda)$ is positive, $f_1(\lambda) \geq m > 0$, and has an r -th derivative satisfying a Hölder condition of order α .

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