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Abstract

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MATHEMATICS

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EMBEDDING THEOREMS FOR WEIGHTED SPACES OF FRACTIONAL ORDER WITH MIXED NORM

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Embedding theorems of the type $M \rightarrow N^*$, where M is an ordinary functional space $(W_{p,\alpha}^l, W_{p,\alpha}^l, B_p^r, H_p^r, S_p^{rH})$, and N is a space with mixed norm, have by now been studied rather well in a number of works ^(6,8,10,11,14). As for embedding theorems of the type $N \rightarrow N$, only for the spaces W_p^l (l integer), H_p^r , S_p^{rH} are there several results ^(9,12,13). In the present note this gap is filled for weighted spaces of fractional order $W_{(p)\alpha}^l(\dot{E}^n)$ with mixed norm. The results obtained are definitive.

Let E^n be the n -dimensional material Euclidean space of points $x = (x_1, \dots, x_n)$; E^m an m -dimensional subspace of the space E^n ;

$$\dot{E}^n = \{x : x_n > 0\}, \quad \dot{E}^{n-m} = \dot{E}^n \setminus E^m, \quad \dot{E}^n = E^{n_1} \times \dots \times E^{n_k}, \quad E^m = E^{m_1} \times \dots$$

$$\dots \times E^{m_\tau},$$

where the decompositions of $\dot{E}^n$ and E^m into subspaces are independent of one another;

$$E^{n_\nu} \cap E^{m_j} = E^{m_{\nu j}}; \quad \dot{E}^{n-m} \cap E^{n_\nu} = E^{n_\nu} \left(\sum_{j=1}^{\tau} m_{\nu j} + n'_\nu = n_\nu \right); \quad \mathbf{p} = (p_1, \dots, p_k), \quad \mathbf{q} = (q_1, \dots, q_\tau),$$

$$\mathbf{l} = (l_1, \dots, l_n), \quad x' = (x_1, \dots, x_{n-1}, 0) = (x'_{n_1}, \dots, x'_{n_k}), \quad x'_{n-m} = (x_{m+1}, \dots, x_{n-1}, 0) = (x'_{n'_1}, \dots, x'_{n'_k})$$

(where x_{n_ν} and $x'_{n'_\nu}$ belong respectively to E^{n_ν} and $E^{n'_\nu}$, $\nu = 1, \dots, k$);

$$\int_{x_1}^{x_1+h\sigma_1} dy_1 \cdots \int_{x_{n-1}}^{x_{n-1}+h\sigma_{n-1}} dy_{n-1} \int_0^{h\sigma_n} [\cdot] dy_n = \int_{\bar{x}}^{x'+h\bar{\sigma}} [\cdot] dy,$$

(h is an arbitrary positive parameter);

$$\left(\int_{x_{n_k}}^{x'_{n_k}+h\bar{\sigma}_{n_k}} dy_{n_k} \cdots \left(\int_{x_{n_1}}^{x'_{n_1}+h\bar{\sigma}_{n_1}} [\cdot]^{p_1} dy_{n_1} \right)^{p_2/p_1} \cdots \right)^{1/p_k} = A_{(x_{n_k};1;h)\bar{\sigma}_{n_k};1}^{(p_k;1)} [\cdot].$$

Further, let $f(y)$ be some smooth finite function in E^n and

$$\|f\|_{L_{(\mathbf{p}),\alpha}^{l_i}(\dot{E}^n)} = \left(\int_0^\infty \frac{dt}{t^{1+p_k l_i}} \|y_n^{\alpha/p_1} \Delta_i(t) f(y)\|_{L_{(\mathbf{p})}(\dot{E}^n)}^{p_k} \right)^{1/p_k} < \infty$$

(where $L_{(\mathbf{p})}(\dot{E}^n)$ is the space with mixed norm; $\Delta_i(t)f(y)$ is the finite difference of first order with step t in the variable x_i);

$$\|f\|_{L_{(\mathbf{p}),\alpha}^l(\dot{E}^n)} = \sum_{i=1}^n \|f\|_{L_{(\mathbf{p}),\alpha}^{l_i}(\dot{E}^n)},$$

$$\|f\|_{W_{(\mathbf{p}),\alpha}^l(\dot{E}^n)} = \|f\|_{L_p \dot{E}^n} + \|f\|_{L_{(\mathbf{p}),\alpha}^l(\dot{E}^n)}.$$

* $E_1 \rightarrow E_2$ denotes the set-theoretic embedding of the class E_1 in the class E_2 with the inequality $\|f\|_{E_2} \leq c\|f\|_{E_1}$ satisfied.

We shall call $L_{(\mathbf{p}),\alpha}^+(E^n)$ and $W_{(\mathbf{p}),\alpha}^+(E^n)$ the closures of the set of smooth finite functions in the corresponding norms.

Theorem 1. If $1 < p_\nu \leq q_j < \infty$ ($\nu = 1, \dots, k$; $j = 1, \dots, \tau$), $1 \leq m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i}; \quad i = 1, \dots, n \right),$$

$$\varepsilon = 1 - \sum_{i=1}^n \sigma_i - \frac{\alpha \sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} > 0$$

(where $\sigma_{\nu\beta}$, $\sigma_\beta^{(j)}$ are numbers corresponding to the coordinate axes of the subspaces E^{n_ν} and E^{m_j}), $f \in W_{(\mathbf{p}),\alpha}^+(E^n)$, then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W_{(\mathbf{p}),\alpha}^l(E^n) \rightarrow L_{(\mathbf{q})}(E^m)$$

holds.

Theorem 2. If $1 < p_\nu \leq q_j < \infty$ ($\nu = 1, \dots, k$; $j = 1, \dots, \tau$), $1 \leq m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i}; \quad i = 1, \dots, n \right),$$

$$\varepsilon \geq \rho_s \sigma_s \quad (0 < \rho_s < 1; \quad s = 1, \dots, m),$$

$$\varepsilon = 1 - \sum_{i=1}^n \sigma_i - \frac{\alpha \sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} > 0, \quad f \in W_{(\mathbf{p}),\alpha}^l(E^n),$$

then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W_{(\mathbf{p}),\alpha}^l(E^n) \rightarrow \bar{L}_{(\mathbf{q})}^\rho(E^m)$$

holds.

From Theorems 1 and 2 follows the validity of the following theorem.

Theorem 3. If $1 < p_\nu \leq q_j < \infty$ ($\nu = 1, \dots, k$; $j = 1, \dots, \tau$), $1 \leq m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i}; \quad i = 1, \dots, n \right),$$

$$\varepsilon = 1 - \sum_{i=1}^n \sigma_i - \frac{\alpha \sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} > 0,$$

$$\varepsilon \geq \rho_s \sigma_s \quad (0 < \rho_s < 1; \quad s = 1, \dots, m), \quad f \in W_{(\mathbf{p}),\alpha}^l(E^n),$$

then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W_{(\mathbf{p}),\alpha}^l(E^n) \rightarrow \bar{W}_{(\mathbf{q})}^\rho(E^m)$$

holds.

Theorem 4. If

$$1 < p_k \leq p_{k-1} \leq \dots \leq p_1 < q_j < \infty \quad (j = 1, \dots, \tau),$$

$$-1 < \alpha < \frac{p_1}{p_k}, \quad 1 \leq m < n, \quad 0 < l_i < 1 \quad (\sigma_i = 1/l_i; \quad i = 1, \dots, n),$$

$$1 - \sum_{i=1}^n \sigma_i - \frac{\alpha \sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \sum_{\nu=1}^k \left(\frac{1}{p_\nu} + \frac{1}{q_j} \right) \sum_{\beta=1}^{m_{\nu j}} \sigma_\beta^{(\nu j)} = 0$$

(where $\sigma_{\nu\beta}$, $\sigma_\beta^{(\nu j)}$ are numbers corresponding to the coordinate axes of the subspaces E^{n_ν} and $E^{m_{\nu j}}$), $f \in L_{(\mathbf{p}),\alpha}^l(E^n)$, then for fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$L_{(\mathbf{p}),\alpha}^l(E^n) \rightarrow L_{(\mathbf{q})}(E^m)$$

holds.

We outline the scheme of the proof, for example, of Theorem 2. From the integral identity of V. P. Il' in ⁽³⁾ we have

$$\|\Delta_s(t)f(x')\|_{L_{(q)}(E^m)} \ll \|\Delta_s(t)F_1(x', h)\|_{L_{(q)}(E^m)} + \|\Delta_s(t)F_2(x', t, h)\|_{L_{(q)}(E^m)} + \|\Delta_s(t)F_3(x', t, h)\|_{L_{(q)}(E^m)}, \quad (1)$$

where

$$F_1(x', h) = \frac{c}{h\omega} \int_0^{h\bar{\sigma}} f(x' + y) \Pi(y, h) dy,$$

$$F_2(x', t, h) = \sum_{i=1}^n C_i \int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+\omega}} \int_0^{\vartheta\bar{\sigma}} dy \int_0^{\vartheta^{\sigma_i} - y_i} \Delta_i(\chi) f(x' + y) R_i(y, \chi, \vartheta) d\chi,$$

$$F_3(x', t, h) = \sum_{i=1}^n C_i \int_{t^{1/\sigma_s}}^h \frac{d\vartheta}{\vartheta^{1+\omega}} \int_0^{\vartheta\bar{\sigma}} dy \int_0^{\vartheta^{\sigma_i} - y_i} \Delta_i(\chi) f(x' + y) R_i(y, \chi, \vartheta) d\chi;$$

each term of the right-hand side of inequality (1) admits the estimate

$$\|\Delta_s(t)F_1(x', h)\|_{L_{(q)}(E^m)} \ll t \|D_s F_1(x_1, \dots, x_s, \dots, x_{n-1}, 0; h)\|_{L_{(q)}(E^m)} \ll$$

$$\ll ct h^{\sum_{\nu=1}^k \frac{1}{p_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} - \sum_{j=1}^n \sigma_j - \sigma_s} \|f\|_{L_{(p)}(\dot{E}^n)}; \quad (2)$$

$$\|\Delta_s(t)F_3(x', t, h)\|_{L_{(q)}(E^m)} \ll t \|D_s F_3(x_1, \dots, x'_s, \dots, x_{n-1}, 0; t, h)\|_{L_{(q)}(E^m)} \ll$$

$$\ll t \sum_{i=1}^n C_i \left\| \int_{t^{1/\sigma_s}}^h \frac{d\vartheta}{\vartheta^{1+\lambda_i-\varepsilon+\sigma_i-\alpha\sigma_n/p_1+\sigma_i\gamma}} \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma-(1/p_k+l_i)} d\chi \int_{x'}^{x'+\vartheta\bar{\sigma}} \Delta_i(\chi) f(y) dy \right\|_{L_{(q)}(E^m)}; \quad (3)$$

$$\begin{aligned} \|\Delta_s(t)F_2(x', t, h)\|_{L_{(q)}(E^m)} &\ll C \left\| \int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+\lambda_i-\varepsilon+\gamma\sigma_i-\alpha\sigma_n/p_1}} \times \right. \\ &\times \left. \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma-(1/p_k+l_i)} d\chi \int_{x'}^{x'+\vartheta\bar{\sigma}} \Delta_i(\chi) f(y) dy \right\|_{L_{(q)}(E^m)} \end{aligned} \quad (4)$$

(in inequalities (3), (4), γ and λ_i are determined from $0 < \gamma \leq 1/p_k + l_i$,

$$\lambda_i = \frac{\sigma_i}{p_k} + \sum_{\nu=1}^k \frac{1}{p_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)}).$$

Using the definition of the space $L_{(q)}^{\rho_s}$, we obtain

$$\begin{aligned} \|f\|_{L_{(q)}^{\rho_s}(E^m)} &\ll \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s q_\tau}} \|\Delta_s(t)f(x')\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} + \\ &+ \left(\int_{h^{\sigma_s}}^\infty \frac{dt}{t^{1+\rho_s q_\tau}} \|\Delta_s(t)f(x')\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} \equiv A[0, h^{\sigma_s}] + A[h^{\sigma_s}, \infty]. \end{aligned}$$

Taking inequalities (1) into account, for $A[0, h^{\sigma_s}]$ we obtain the estimate

$$\begin{aligned} A[0, h^{\sigma_s}] &\ll \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+q_\tau \rho_s}} \|\Delta_s(t)F_1(x', h)\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} + \\ &+ \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+p_s q_\tau}} \|\Delta_s(t)F_2(x', t, h)\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} \end{aligned}$$

$$+ \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+p_s q_\tau}} \|\Delta_s(t) F_3(x', t, h)\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} \equiv A_1^{(s)} + A_2^{(s)} + A_3^{(s)}.$$

We estimate $A_1^{(s)}, A_2^{(s)}, A_3^{(s)}$: on the basis of inequality (2)

$$A_1^{(s)} \ll ch^{\sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_\nu \beta + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} - \sum_{j=1}^n \sigma_j - \rho_s \sigma_s} \|f\|_{L_{(p)}(E^n)}, \quad (5)$$

taking into account inequality (4),

$$A_2^{(s)} \ll ch^{\varepsilon - \rho_s \sigma_s} \sum_{i=1}^n \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\nu \sigma_i q_\tau / \sigma_s - \xi q_\tau / \sigma_s}} \left\| \left(\int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+p_k \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} + \xi p_k}} \right. \right. \right. \\ \left. \left. \left. \times \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma p_k - (1+p_k l_i)} d\chi \left(A_{(x_{n_k;1}; \vartheta^{\bar{\sigma}_{n_k;1}})}^{(p_k;1)} \left[y_n^{a/p_1} \Delta_i(\chi) f(y) \right] \right)^{p_k} \right)^{1/p_k} \right\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau}$$

(where ξ is, for the time being, an arbitrary positive number).

Having made simple estimates for the norm in the curly bracket, we obtain

$$A_2^{(s)} \ll ch^{\varepsilon - \rho_s \sigma_s} \|f\|_{L_{(p), \alpha}^1(E^n)}. \quad (6)$$

Similarly we obtain

$$A_3^{(s)} \ll ch^{\varepsilon - \rho_s \sigma_s} \|f\|_{L_{(p), \alpha}^1(E^n)}. \quad (7)$$

It is easy to note that

$$A[h^{\sigma_s}, \infty] \ll ch^{-\sigma_s \rho_s} \|f\|_{\dot{W}_{(p), \alpha}^t(E^n)}. \quad (8)$$

From the estimates given above, taking into account inequalities (5), (6), (7), (8) and setting $h = 1$, we obtain the assertion of the theorem.

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