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Abstract

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MECHANICS

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ON AN INVERSE PROBLEM IN DIFFRACTION THEORY

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The scalar problem of diffraction of a plane wave $e^{ikr \cos \gamma}$ ($\cos \gamma = \cos \theta \cos \alpha + \sin \theta \sin \alpha \cos(\varphi - \beta)$; r, θ, φ are spherical coordinates; α, β are angles determining the direction of the incident wave) by a sphere of radius r_0 with center at the point $r = 0$ and with a variable admittance g , prescribed on the surface of the sphere, of the form

$$g(\theta, \rho) = \sum_{l=0}^{N-1} g_l(\rho) P_l(\cos \theta) \quad (\operatorname{Re} g \geq 0) \quad (1)$$

(N is a given positive integer; $P_l(\cos \theta)$ are Legendre polynomials; $g_l(\rho)$ are rational functions of the parameter $\rho = kr_0$, which will subsequently be chosen in a special way) is formulated as follows: in the region $r > r_0$ outside the sphere it is required to find a function $p(r, \theta, \varphi, \alpha, \beta, N, k)$, meromorphically dependent on the parameter k , satisfying, with respect to the variables r, θ, φ , the Helmholtz equation

$$\Delta p + k^2 p = 0 \quad (p \sim e^{-i\omega t}) \quad (2)$$

and the absorption condition ⁽¹⁻³⁾

$$|p - e^{ikr \cos \gamma}| < \infty \quad \text{for} \quad \operatorname{Im} k > |\operatorname{Re} k|, \quad (3)$$

and, on the surface of the sphere, the boundary condition

$$\partial p / \partial r + ikgp = 0 \quad \text{for} \quad r = r_0. \quad (4)$$

For given coefficients $g_l(\rho)$ and number N , the formulated direct diffraction problem (1), (2), (3), (4) has a unique ^(2,3) solution $p(r, \theta, \varphi, \alpha, \beta, N, k)$.

Denote by $p_N(\rho, \alpha)$ the value of the function $p(r, \theta, \varphi, \alpha, \beta, N, k)$ at the point $r = r_0$, $\theta = 0$ of the surface of the sphere:

$$p_N(\rho, \alpha) = p(r_0, 0, \varphi, \alpha, \beta, N, k). \quad (5)$$

Owing to the axial symmetry of the domain and of the boundary condition, it does not depend on the angles φ, β .

In the present note a formulation is proposed of the inverse diffraction problem, consisting in the determination of such coefficients $g_l(\rho)$ entering into formula (1) that, as $\rho \rightarrow 0$, the ratio

$$f(\alpha, \rho) = p_N(\rho, \alpha)/p_N(\rho, 0)$$

approximates as accurately as possible a certain optimal function $f_N(\alpha)$.

As such a function we take

$$f_N(\alpha) = (N+1)^{-2} \sum_{n=0}^N (2n+1) P_n(\cos \alpha), \quad (6)$$

having, at the single point of the closed interval $0 \leq \alpha \leq \pi$, an absolute maximum (in modulus) $f_N(0) = 1$. The coefficients in the right-hand side of (6) are chosen so that, for the given natural number N and under the condition of uniqueness of the absolute maximum $f_N(0) = 1$, the functional of Yu. M. Sukharevskii ⁽⁴⁾

$$K_N = 2 \int_0^\pi |f_N(\alpha)|^2 \sin \alpha \, d\alpha \quad (7)$$

takes, for the function (6), the greatest possible value (in comparison with the values of this functional for functions of the form (6) with other real or complex coefficients)

$$K_N = (N+1)^2. \quad (8)$$

The condition that the absolute maximum of $f_N(\alpha)$ be attained at the single point $\alpha = 0$ is essential here. If two points are allowed, for example $f_N(0) = f_N(\pi) = 1$, then instead of (6), (8) we have

$$f_N(\alpha) = (N+1)^{-1} (2N+1)^{-1} \sum_{n=0}^N (4n+1) P_{2n}(\cos \alpha), \quad (6^*)$$

$$K_N = (N + 1)(2N + 1). \quad (8^*)$$

Here we shall confine ourselves to an example of the exact solution of such an optimal inverse problem for the simplest case $N = 1$, when, according to (1),

$$g(\theta, \rho) = g_0(\rho). \quad (9)$$

The solution of the direct diffraction problem (2), (3), (4), (9) has the form

$$p(r, \theta, \varphi, \alpha, \beta, 1, k) = \sum_{n=0}^{\infty} i^n (2n + 1) \left[j_n(kr) - \frac{j'_n(\rho) + ig_0 j_n(\rho)}{h'_n(\rho) + ig_0 h_n(\rho)} h_n(kr) \right] \times \\ \times \left[P_n(\cos \theta) P_n(\cos \alpha) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \alpha) \cos m(\varphi - \beta) \right],$$

where $j_n(\rho) = \sqrt{\pi/2\rho} J_{n+1/2}(\rho)$; $h_n(\rho) = \sqrt{\pi/2\rho} H_{n+1/2}^{(1)}(\rho)$ are spherical Bessel functions. In particular, according to (5), for the point $r = r_0$, $\theta = 0$ on the surface of the sphere we obtain

$$p_1(\rho, \alpha) = \sum_{n=0}^{\infty} i^n (2n + 1) P_n(\cos \alpha) \left[j_n(\rho) - \frac{j'_n(\rho) + ig_0 j_n(\rho)}{h'_n(\rho) + ig_0 h_n(\rho)} h_n(\rho) \right] \\ = \sum_{n=0}^{\infty} \frac{i^{n+1} (2n + 1) P_n(\cos \alpha)}{\rho^2 [h'_n(\rho) + ig_0 h_n(\rho)]}.$$

Hence, using the representation

$$h_n(\rho) = \frac{e^{i\rho}}{i^{n+1} \rho} \sum_{m=0}^n \frac{(-)^m}{m!} \frac{(n+m)!}{(n-m)!} \frac{1}{(2i\rho)^m},$$

we find the relation

$$f(\alpha, \rho) = \frac{p_1(\rho, \alpha)}{p_1(\rho, 0)} = \frac{\sum_{n=0}^{\infty} (2n + 1) c_n(\rho) (2i\rho)^n P_n(\cos \alpha)}{\sum_{n=0}^{\infty} (2n + 1) c_n(\rho) (2i\rho)^n}, \quad (10)$$

where it is denoted

$$\frac{1}{c_n(\rho)} = \sum_{m=0}^n \frac{(-)^m}{m!} \frac{(2n-m)!}{(n-m)!} (n-m+1 - i\rho g_0 - i\rho) (2i\rho)^m. \quad (11)$$

If the admittance g_0 were a fixed complex number, then from (10), (11) it would follow that

$$f(\alpha, \rho) = 1 + O(\rho) \quad \text{as } \rho \rightarrow 0 \quad (P_0(\cos \alpha) \equiv 1).$$

However, if one chooses the coefficient g_0 in the form of a function of the parameter ρ , which in the present case is the solution of the inverse problem posed,

$$g_0(\rho) = 1 - i \cdot 2/\rho, \quad (12)$$

then from (10), (11) we obtain:

$$f(\alpha, \rho) = f_1(\alpha) + O(\rho) \quad \text{as } \rho \rightarrow 0, \quad (13)$$

which in the limit $\rho \rightarrow 0$ coincides with the optimal expression

$$f_1(\alpha) = \frac{1}{4} \sum_{n=0}^1 (2n+1) P_n(\cos \alpha) = \frac{1 + 3 \cos \alpha}{4},$$

given by formula (6) for $N = 1$, so that, according to (8), the maximum possible value (under an absolute maximum, at the single point $\alpha = 0$) of Sukharevsky's functional (7) is attained:

$$K_1 = 4. \quad (14)$$

If, however, an absolute maximum is allowed at two points $f_1(0) = f_1(\pi) = 1$, the solution of the inverse problem

$$g_0 = -i(3/\rho - \rho) \quad (15)$$

leads, according to (10), (11), (6), (8), to the function optimal for $N = 1$,

$$f(\alpha, 0) = f_1(\alpha) = (5 \cos^2 \alpha - 1)/4, \quad (16)$$

which gives the maximum value of the functional (7)

$$K_1 = 6. \quad (17)$$

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Note: Figure translations are in progress. See original paper for figures.

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