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Abstract

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MATHEMATICS

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REPRESENTING SPACES FOR THE K -FUNCTOR WITH COEFFICIENTS

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This paper describes the connection between the Adams-Toda homotopy operations (see ⁽¹⁾, Theorem 1.7) and the differentials in the Atiyah-Hirzebruch spectral sequence (see ⁽²⁾). The results obtained make it possible to describe the homotopy type of representing spaces for K -theory with coefficients in the rings Z_p and $Z[t]/(t+1)^p - 1$, where p is an odd prime.

We shall consider only complex K -theory. The Atiyah-Hirzebruch spectral sequence for a complex X will be denoted by $SP(X)$ for the K -functor and by $SP(X; Z_p)$ for the K -functor mod p . We note that by a cellular complex we shall mean a cellular complex with finite skeleta.

I. Lemma 1. *For any $n > 2p - 3$ there exists a sequence of mappings*

$$A(n) = \{\alpha_s; \alpha_s : S_p^{n+2s(p-1)-1} \rightarrow S_p^n\}$$

such that: a)

$$\alpha_s = (\Sigma^{2s(p-1)} \alpha_1) \circ \dots \circ \alpha_1;$$

b) for the complex

$$W_s = S_p^n \cup_{\alpha_s} D_p^{n+2s(p-1)},$$

in the spectral sequences $SP(W_s)$ and $SP(W_s, Z_p)$ the differential $d_{2s(p-1)+1}$ acts nontrivially. Here

$$S_p^n = S^n \cup_p D^{n+1}.$$

Proof. Consider a mapping $f : S_p^{2p} \rightarrow S^3$ with Hopf invariant 1 mod p (see ⁽³⁾). It is easy to show that for $n > 2p - 3$ the mapping $\Sigma^{n-3} f$ decomposes into the composition

$$\pi \circ \alpha_1 : S_p^{n+2p-3} \rightarrow S_p^{n-1}.$$

Since

$$d_{2p-1} = e_{2,1},$$

we have $K^*(W_1) = 0$, and the mapping

$$\alpha_1^* : K^*(S_p^{n-1}) \rightarrow K^*(S_p^{n+2p-3})$$

is an isomorphism. Consequently, for every s the mapping α_s^* is an isomorphism and $K^*(W_s) = 0$. The lemma is proved.

Theorem 1. *For any $n > 2p - 3$ there exists a cellular complex $Y(Z_p, n)$ such that:*

a)

$$\pi_q(Y(Z_p, n)) = \begin{cases} Z_p, & q \equiv n \pmod{2(p-1)}, \\ 0, & q \not\equiv n \pmod{2(p-1)}; \end{cases}$$

b) *the generating element*

$$u_n \in H^n(Y(Z_p, n), Z_p)$$

is a cycle of all differentials in $SP(Y(Z_p, n); Z_p)$.

The complex $Y(Z_p, n)$ has the following properties:

1) *All cellular complexes Y satisfying conditions a) and b) are homotopy equivalent to the complex $Y(Z_p, n)$.*

2) *A mapping*

$$f : Y(Z_p, n) \rightarrow Y(Z_p, n)$$

is a homotopy equivalence if and only if $f^*(u_n) \neq 0$.

Proof. From the sequence $A(n) = \{\alpha_s\}$ one can construct a sequence of cellular complexes $\{Y_s\}$ as follows. As Y_0 take the Eilenberg-Mac Lane space $K(Z_p, n)$, $n > 2p - 3$; as Y_1 , take the space of the fibration

$$Y_1 \xrightarrow{k(Z_p, n+2(p-1))} Y_0$$

with Postnikov invariant $e_{2,1}$. Denote by a_1 the generator of the group $H^n(Y_1; Z_p)$, and by b_1 the generator of the group $\pi_n^p(Y_1)$. By construction, in $SP(Y_1; Z_p)$ the element

$$a_1 \in E_{2p-1}^{n, 2t}$$

is a cycle of the differential d_{2p-1} , and therefore the generator of the group $\pi_{n+2(p-1)}(Y_1)$ can be decomposed into the composition

$$S^{n+2(p-1)} \subset S_p^{n+2(p-1)} \xrightarrow{\alpha_1} S_p^n \subset Y_1.$$

According to Theorem 2 of (5), the element

$$a_1 \in E_{2p-1}^{n, 2t}$$

is a cycle of the differentials d_r for $r < 4(p-1) + 1$

The embedding $b_1 : S_p^n \subset Y_1$ extends to a map $\tilde{b}_1 : W_2 \rightarrow Y_1$, and since, by Lemma 1, the differential $d_{4(p-1)+1}$ acts nontrivially in the complex W_2 , we obtain that $d_{4(p-1)+1}(a_1) \neq 0$. Thus, in the group

$$E_{4(p-1)+1}^{n-4(p-1)+1, 2t-4(p-1)} \subset SP(Y_1, Z_p)$$

an element is marked. Take as Y_2 the fibration space

$$Y_2 \xrightarrow{K(Z_p, n+4(p-1))} Y_1$$

with Postnikov invariant $c_2 \in H^{n+4(p-1)+1}(Y_1, Z_p)$, where c_2 is some representative of the element $d_{4(p-1)+1}(a_1)$ in cohomology. Denote by $a_2 \in H^n(Y_2, Z_p)$ and $b_2 \in \pi_n^p(Y_2)$ the generating elements. Using the complex W_2 , we obtain that the generating group $\pi_{n+4(p-1)}(Y_2) = Z_p$ decomposes into the composition

$$S^{n+4(p-1)} \subset S_p^{n+4(p-1)} \xrightarrow[b_2]{a_2} S_p^n \subset Y_2.$$

Continuing the construction, we obtain a sequence of complexes

$$\{Y_i, Y_i \xrightarrow{K(Z_p, n+2i(p-1))} Y_{i-1}\}, \quad i \geq 0.$$

The $(n + 2i(p-1) - 1)$ -dimensional skeleta of the spaces Y_i and Y_{i-1} can be identified for any i , and therefore we have a sequence of embedded cell complexes

$$[Y_0]_{n+2(p-1)-1} \subset [Y_1]_{n+4(p-1)-1} \subset \cdots \subset [Y_i]_{n+2(i+1)(p-1)-1} \subset \cdots.$$

Put

$$Y(Z_p, n) = \varinjlim [Y_i]_{n+2(i+1)(p-1)-1}.$$

It is clear that $Y(Z_p, n)$ is a cell complex. Denote by $u_n \in H^n(Y(Z_p, n), Z_p)$ a generator of the group.

By construction, u_n is a cycle of all differentials in $SP(\cdot, Z_p)$, and moreover, for any cell complex X and element $x \in H^n(X, Z_p)$ that is a cycle of all differentials in $SP(\cdot, Z_p)$, there exists a continuous map $f : X \rightarrow Y(Z_p, n)$ such that $f^*u_n = x$. Using now the fact that, for any s , the generator of the group $\pi_{n+2s(p-1)}(Y(Z_p, n))$ decomposes into the composition

$$S^{n+2s(p-1)} \rightarrow S_p^{n+2s(p-1)} \xrightarrow{\alpha_s} S_p^n \subset Y(Z_p, n),$$

we obtain the proof of properties 1) and 2) of the space $Y(Z_p, n)$. The theorem is proved.

Since the spectral sequence $SP(Y, (Z_p, n)Z_p)$ converges strongly ⁽⁴⁾, in the group $k^n(Y(Z_p, n)Z_p)$ there exists an element ζ_n of filtration n .

Lemma 2. a) For any $n > 2p - 3$ there exists a homotopy equivalence

$$\chi_n : \Omega Y(Z_p, n+1) \xrightarrow{\simeq} Y(Z_p, n).$$

b) There exists a homotopy equivalence

$$\chi : \Omega^{2(p-1)} Y(Z_p, 2(p-1)) \rightarrow Z_p \times Y(Z_p, 2(p-1)).$$

Proof. Let K_p be a representing space for the K -functor $\text{mod } p$. Denote by $K_p(2n)$ the $2n$ -decreasing space for the complex K_p , and by $K_p(2n-1)$ the space $\Omega K_p(2n)$. According to the classifying properties of the spaces $K_p(n)$, there exists a map $f : Y(Z_p, n+1) \rightarrow K_p(n+1)$ such that $f^*(\eta_{n+1}) = \zeta_{n+1}$, where η_{n+1} is the universal element. Consider the map

$$\Omega f : \Omega Y(Z_p, n) \rightarrow \Omega K_p(n).$$

Since the homomorphism

$$(\Omega f)^* : H^n(\Omega K_p(n+1), Z_p) \rightarrow H^n(\Omega Y(Z_p, n+1), Z_p)$$

is nonzero, the generating element

$$c \in H^n(\Omega Y(Z_p, n+1), Z_p)$$

is, obviously, a cycle of all differentials in $SP(_, Z_p)$. Applying now Theorem 1, we obtain that the spaces $\Omega Y(Z_p, n+1)$ and $Y(Z_p, n)$ are homotopy equivalent. Part a) is proved. Part b) is proved analogously.

Corollary 1. For any $n > 2p - 3$ there exists a homotopy equivalence λ :

$$\prod_{k=0}^{p-1} Y(Z_p, n+2k) \rightarrow K_p(n).$$

Put

$$h_p^{-q}(X, Y) = [S^q(X/Y), Z_p \times Y(Z_p, 2(p-1))]_*, \quad q > 0.$$

The homotopy equivalence

$$\chi : \Omega^{2(p-1)} Y(Z_p, 2(p-1)) \rightarrow Z_p \times Y(Z_p, 2(p-1))$$

makes it possible to define the groups h_p^{-q} also for negative-

q . By the standard method it is easy to verify that the homotopy functor h_p^* defines an extraordinary cohomology theory $(2(p-1))$ -periodic. Applying Lemma 2 and Corollary 1, we obtain the theorem:

Theorem 2. On the category of cellular complexes there exists a $2(p-1)$ -periodic cohomology theory $h_p^* = \sum h_p^q$ such that for any pair of cellular complexes (X, Y) there is a natural isomorphism

$$\tau_p : \sum_{i=0}^{p-1} h_p^{-2i}(X, Y) \xrightarrow{\approx} k^0(X, Y; \mathbb{Z}_p).$$

II. Let L_p^∞ be the infinite-dimensional lens space. Denote by L^n the $2n$ -dimensional skeleton of the space L_p^∞ . Denote the group $k^0(L^n)$ by A_n and put, as usual, $\tilde{k}^0(X; A_n) = k^0(X \# L^n)$. Since $L^1 = S_p^1$, it follows that $k^*(; A_1) = k^*(; \mathbb{Z}_p)$. For any n the equality $L^{n+1}/L^n = S_p^{2n+1} = S^{2n}S_p^1$ holds, which makes it possible to generalize Theorem 2.

Theorem 3. *On the category of cellular complexes, for every $n > 0$ there exist $2(p-1)$ -periodic cohomology theories $h_{p,n}^*$ and natural transformations $\tau_{p,n}^0 : h_{p,n}^0 \rightarrow k^0(; A_n)$, satisfying the following conditions:*

a) *for any pair of cellular complexes there is an isomorphism*

$$\tau_p : \sum_{i=0}^{p-1} h_{p,n}^{-2i}(X, Y) \rightarrow k^0(X, Y; A_n),$$

b) *there are transformations $p_n : h_{p,n} \rightarrow h_{p,n-1}$ such that $\pi_n^* \tau_{p,n} = \tau_{p,n-1} p_n$, where the projection $\pi_n : k^*(; A_n) \rightarrow k^*(; A_{n-1})$ is induced by the inclusion $\pi_n : L^{n-1} \rightarrow L^n$.*

Proof. Denote by V^n the space of continuous mappings $\{L^n \rightarrow BU\}_*$ preserving the base point. Since L^n is a finite complex, the space V^n is a representing space for the $k^*(; A_n)$ -theory. Suppose that for all $q \leq n$ cellular complexes Y_q and mappings $p_q : Y_q \rightarrow Y_{q-1}$, $\tau_q^0 : Y_q \rightarrow V^q$ have been constructed such that $\tau_{q-1}^0 p_q \simeq f_{q-1} \tau_q^0$ and τ_q^0 induces an isomorphism of the groups $\pi_{2s(p-1)}$, $s \geq 0$. (For $n = 1$, $V^1 = K_p$, therefore as Y_1 one may take $Y(\mathbb{Z}_p, 2(p-1))$.) We construct the complex Y_{n+1} and mappings p_n and τ_{n+1}^0 satisfying these conditions. Consider the segment of the Puppe sequence

$$(\pi_n : L^n \subset L^{n+1}) : S^{2n+1}S_p^1 \rightarrow S^2L^n \rightarrow S^2L^{n+1}.$$

There is a Serre fibration $g : \Omega^2V^n \rightarrow \Omega^{2n+1}K_p$ with fiber $F^a \simeq \Omega^2V^{n+1}$, and moreover $i = \Omega^2f_n \cdot a$, where $i : F \rightarrow \Omega^2V^n$ is the inclusion of the fiber. Using standard homotopy technique, it is easy to show that F is homotopy equivalent to the fibration space $F' \rightarrow \Omega^2V^n$, induced by the universal Serre fibration $* = p\Omega^{2n+1}K_p \rightarrow \Omega^{2n+1}K_p$ by means of the mapping g . Consider the composition of mappings

$$\varphi : \Omega^2(Y_n)_0 \xrightarrow{\tau_n^0} \Omega^2(V^n)_0 \xrightarrow{g} \Omega^{2n+1}(K_p)_0 \xrightarrow{\Omega^{2n+1}} \Omega^{2n+1}Y(\mathbb{Z}_p, 2p-1)$$

and denote by Y_{n+1} the fibration space $Y_{n+1} \rightarrow \Omega^2(Y_n)_0$, induced by the fibration $* \rightarrow \Omega^{2n+1}Y(\mathbb{Z}_p, 2p-1)$ by means of the mapping φ . It follows directly from the construction that there exist mappings p_n and τ_{n+1}^0 satisfying the required conditions. The induction step is complete.

Put $h_{p,n}^{-q}(X, Y) = [S^{qX}/Y, A_n \times Y]$. It is easy to verify that the cohomology theory $h_{p,n}^*$ and the transformation $(\tau_n^0)^* : h_{p,n} = k^0(; A_n)$ satisfy all the conditions of the theorem.

III. We now consider the K -functor with coefficients in the ring $k^*(L_p^\infty)$. Since $H^*(L_p^\infty; Q) = 0$, we have $k^*(L_p^\infty) = \mathcal{K}^*(L_p^\infty)$ (see (4)). It is easy to show that $\mathcal{K}^1(L_p^\infty) = 0$ and $\mathcal{K}^0(L_p^\infty) = \mathbb{Z}[t]/(t+1)^p - 1$, where $t = \zeta - 1$, and ζ is the canonical line bundle over L_p^∞ . Denote the ring

$\mathbb{Z}[t]/(t+1)^p - 1$ by A . The ring A is additively isomorphic to the direct sum of $(p-1)$ copies of the ring of integral p -adic numbers, and therefore in questions where consideration of the K -functor with p -adic coefficients is required, one may use the functor $k(; A)$ (see (4), § 8). In (4) it is shown that if X is a finite complex, then there is an isomorphism

$$k^*(X; A) \simeq K^*(X) \otimes_{\mathbb{Z}} A.$$

For an arbitrary complex X there exists an exact sequence relating the groups $k^*(X; A)$ and $K^*(X)$. From the results of Section II it follows:

Theorem 4. *On the category of cellular complexes there exists a $2(p-1)$ -periodic cohomology theory $\mathcal{H}_p$ and a natural transformation $\tau_p^0 : \mathcal{H}_p^0 \rightarrow k^0(; A)$, which for any pair of complexes (X, Y) induces an isomorphism*

$$\tau_p : \sum_{i=0}^{p-1} \mathcal{H}_p^{-2i}(Z, Y) \rightarrow k^0(X; A).$$

Proof. From Theorem 3 it follows that for any pair of cellular complexes (X, Y) and any integer q an inverse sequence of groups $\{h_{p,n}^q(X, Y)\}_{n>0}$ is defined. Since $H^*(L^n; Q) = 0$, for any cellular complex X there is an isomorphism

$$k^q(X; A_n) \simeq \lim_{\leftarrow} k^q(X_l; A_n),$$

where X_l is the l -dimensional skeleton of the complex X . For every l the group $k^q(X_l; A_n)$ is finite; hence the group $k^q(X; A_n)$ is profinite. It follows that the group $h_{p,n}^q(X) \subset k^q(X; A_n)$ is also profinite. Put

$$\mathcal{H}_p^q(X, Y) = \lim_{\leftarrow} h_{p,n}^q(X, Y).$$

Using (4), § 3, we obtain that the functor $\mathcal{H}_p^* = \sum \mathcal{H}_p^q$ defines a cohomology theory on the category of cellular complexes. For any complex X there is an isomorphism

$$k^*(X \# L_p^\infty) \simeq K^*(X \# L_p^\infty)$$

(see ⁽⁴⁾). Clearly,

$$k^*(X \# L_p^\infty) \simeq \varprojlim k^*(X \# L^n).$$

Thus, the sequences of transformations $\{\tau_{p,n}^0\}$ and $\{\tau_{p,n}\}$ induce transformations $\tau_p^* : \mathcal{H}_p^0 \rightarrow k^0(\cdot; A)$ and

$$\tau_p : \sum \mathcal{H}_p^{-2i}(X, Y) \rightarrow k^0(X; A).$$

The theorem is proved.

IV. In Section II we introduced a sequence of fibrations in the sense of Serre

$$\{f_n : V \rightarrow V^n, f_n^{-1}(*) \simeq \Omega^{2n} K_p\}.$$

Put $V = \varprojlim V^n$. Clearly, for every n the projection $\pi_n : V \rightarrow V^n$ is a fibration in the sense of Serre. All V^n are H -spaces and f_n are homomorphisms; therefore V is also an H -space.

Theorem 5. *For every finite complex X there is an isomorphism $k^0(X; A) = [X, V]_*$, where $A = k^*(L_p^\infty)$.*

Proof. It is easy to verify that there is an epimorphism

$$\alpha : [X, V]_* \rightarrow \varprojlim [X, V^n] = k^0(X, A).$$

Introduce in the space V the filtration $\{V_n\}$, putting

$$V_0 = V, \quad V_n = \pi_n^{-1}(V^n), \quad * \in V^n.$$

Denote by $i_n : V_n \subset V$ and $i_{n,k} : V_n \subset V_k$ the natural embeddings. Clearly, all V_n are fibrations in the sense of Serre.

$$V_n = \varprojlim V_n^k.$$

Let

$$\pi_n^k : V_n \rightarrow V_n^k$$

be the corresponding projections of fibrations.

Lemma 3. *Let X be a cellular complex and let $\{\varphi_n : X \rightarrow V_n\}$ be a sequence of maps such that $\varphi_n \simeq i_{n+1,n} \circ \varphi_{n+1}$. Then the map $\varphi = \varphi_0 : X \rightarrow V$ is homotopic to a map to a point.*

Now let X be a finite complex and $x \in \ker \alpha$. Put $\varphi_0 = x$, and, using the fact that for every finite complex the group $[X, \Omega^{2n} K_p]$ is finite, we can by induction construct a sequence of maps satisfying the lemma. The theorem is proved.

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