

# THE MAXIMUM PRINCIPLE IN THE CLASS OF VARIATIONS SMALL IN ABSOLUTE VALUE, FOR OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS OF EQUALITY AND INEQUALITY TYPE

A. Ya. Dubovitskii, A. A. Milyutin

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**Abstract**

**Full Text**

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**MATHEMATICS**

**A. Ya. Dubovitskii, A. A. Milyutin**

# THE MAXIMUM PRINCIPLE IN THE CLASS OF VARIATIONS SMALL IN ABSOLUTE VALUE, FOR OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS OF EQUALITY AND INEQUALITY TYPE

*(Presented by Academician L. S. Pontryagin on 23 V 1969)*

In this note the following problem is solved. In the class of variations small in absolute value, find necessary conditions for a minimum of the functional

$$I = \int_0^1 F(x(t), u(t), t) dt, \quad x \in E^n, \quad u \in E^r,$$

under the constraints

$$dx/dt = f(x, u, t), \quad x(0) = x_0, \quad x(1) = x_1, \quad g_k(x, u) = 0, \quad k = 1, \dots, k, \quad x, u \in Q_i,$$

$$i = 1, \dots, I.$$

**Assumptions.**  $x(t), u(t)$  are bounded measurable functions.

$F(x, u, t), f(x, u, t)$ : a) are bounded together with their first derivatives with respect to  $x, u$  on every bounded set of  $x, u, t$ ; b) the derivatives with respect to  $x, u$  are uniformly continuous on every bounded set of  $x, u, t$ .

$g_k(x, u)$ : a) are continuously differentiable; b)  $g_{ku}$  are linearly independent if  $g_k(x, u) = 0$ ; c)  $k < r$ ,  $Q_i$  is an open set,  $\kappa_i$  is the boundary of  $Q_i$ .

a) For every point  $x, u \in \kappa_i$  there exists an open convex cone  $\Omega_i(x, u)$  in the space  $x, u$  such that:

- 1) if  $\bar{x}, \bar{u} \in \bar{\Omega}_i(x, u)$ , then  $x + \varepsilon \bar{x}, u + \varepsilon \bar{u} \in \bar{Q}_i$ ;
- 2) if  $\bar{x}, \bar{u} \notin \bar{\Omega}_i(x, u)$ , then  $x + \varepsilon \bar{x}, u + \varepsilon \bar{u} \notin \bar{Q}_i$

for all sufficiently small  $\varepsilon > 0$  (the bar denotes closure); b) the set  $x, u, n : x, u \in \mathcal{N}_i, n \in N_i(x, u)$ , where  $N_i(x, u)$  is the set of exterior normals to  $\Omega_i(x, u)$ , is closed in the space  $x, u, n$ .

Let  $x^0, u^0 | t$  be an optimal trajectory. If for arbitrary  $\varphi_k(t), \xi$  there exist such  $\bar{x}(t), \bar{u}(t)$  that

$$d\bar{x}/dt = f_x \bar{x} + f_u \bar{u}, \quad \bar{x}(0) = 0, \quad \bar{x}(1) = \xi,$$

$$g_{kx} \bar{x}(t) + g_{ku} \bar{u}(t) = \varphi_k(t)$$

(the derivatives are taken at the points  $t; x^0 u^0 | t$ ), then we shall say that the nondegeneracy condition is fulfilled for the trajectory. We shall assume that this condition is fulfilled for the trajectory  $x^0, u^0 | t$ .

**Variations.** In view of the nondegeneracy condition of the trajectory  $x^0, u^0 | t$ , the set  $L$  of variations  $\bar{x}(t), \bar{u}(t)$  admissible with respect to the equality constraints is determined by the following conditions:

$$d\bar{x}/dt = f_x \bar{x} + f_u \bar{u}, \quad \bar{x}(0) = \bar{x}(1) = 0, \quad g_{kx} \bar{x} + g_{ku} \bar{u} = 0.$$

Put  $n \in N_{i\delta}(x, u)$ ,  $x, u \in \bar{Q}_i$ , if  $n \in N_i(x', u')$ ,  $x', u' \in \mathcal{N}_i$ ,  $\|x, u - x', u'\| \leq \delta$ ;  $\bar{N}_{i\delta}(x, u)$  is the convex closure of  $N_{i\delta}(x, u)$ . Put  $n, t \in \Pi_{i\delta}$ , if  $N_{i\delta}(x^0, u^0 | t) \neq \emptyset$  and  $n \in \bar{N}_{i\delta}(x^0, u^0 | t)$ .

If for some  $\delta > 0$

$$n_x \bar{x}(t) + n_u \bar{u}(t) \leq -c < 0, \quad n, t \in \Pi_{i\delta},$$

then the variation  $\bar{x}, \bar{u} | t$  is admissible with respect to the constraint  $x, u \in \bar{Q}_i$ , and conversely. The set of admissible variations forms a nonempty open cone  $\Omega_i$ . The set  $\Omega_0$  of forbidden variations is defined by the inequality

$$\int_0^1 (F_x \bar{x} + F_u \bar{u}) dt < 0.$$

The stationarity condition consists in the intersection  $\Omega_0, L, \dots, \Omega_i, \dots$  being empty.

I. To translate this condition into the language of metric properties of the trajectory  $x^0, u^0 | t$ , we shall use Theorems 1 and 2 and the lemma of the present note.

**Theorem 1.** Let  $Z, X, Y$  be normed spaces, with  $X = Y^*$ ,  $Z = X^*$ . In  $X$  a system of cones  $\Omega_0, \dots, \Omega_n$ ,  $\Omega$  is given, where  $\Omega_0, \dots, \Omega_n$  are open convex cones,  $\Omega$  is a convex cone;  $x_0 \in \Omega_0, \dots, x_n \in \Omega_n$  is a certain system of elements.

Suppose that there are convex cones  $\Omega_{iy}^* \subseteq \Omega_i^*$ ,  $\Omega_y^* \subseteq \Omega^*$ , weakly dense in  $\Omega_0^*, \dots, \Omega^*$  and consisting of elements of the space  $Y^*$ .

Let  $B_\varepsilon$  be the set of systems  $(y_0, \dots, y_n, y)$  such that  $y_i \in \Omega_{iy}^*$ ,

$$y \in \Omega_y^*, \quad \|y_0 + \dots + y_n + y\| \leq \varepsilon, \quad \sum_0^n y_i(x_i) = 1.$$

Let  $B$  be the set of systems  $z_0, \dots, z_n, z$  that are solutions of the Euler equation relative to the system of cones  $\Omega_0, \dots, \Omega_n, \Omega$  (i.e.  $z_i \in \Omega_i^*$ ,  $z \in \Omega^*$ ,

$$\sum_0^n z_i + z = 0$$

) such that

$$\sum_0^n z_i(x_i) = 1.$$

Then

$$B = \bigcap_{\varepsilon > 0} \overline{B_\varepsilon}$$

(closure in the weak sense).

In our case the space  $X = L_\infty^{n+r}$ . The space  $Y = L_1^{n+r}$ .

**Lemma.** Let  $r_1(t), \dots, r_j(t), \dots$  be a sequence of measurable vector functions ( $r \in E^n$ ) possessing the following properties:

- 1)  $\int |r_j| dt \leq \text{const};$
- 2) for any natural  $N \geq 0$  there exist:
  - a)

$$r_N^0(t) = \lim_j \text{cl}(L_1) N r_j(t),$$

where

$$Nr = \begin{cases} r, & |r| \leq N, \\ \frac{N}{|r|} r, & |r| > N; \end{cases}$$

b)

$$\lambda_N^0(t) = \lim_j \text{cl}(L_1) N \lambda_j(t),$$

where

$$\lambda_j(t) = |r_j(t)|.$$

Put

$$\lambda^0(t) = \lim_{N \rightarrow \infty} \lambda_N^0(t), \quad r^0(t) = \lim_{N \rightarrow \infty} r_N^0(t)$$

(the convergence takes place in  $L_1$ ).

Then there exists an integer-valued function  $N(j) \geq 0$  such that

1)

$$\lim_j \text{cl}(L_\infty) N(j) \lambda_j(t) = \lambda^0(t),$$

$$\lim_j \text{cl}(L_\infty) N(j) r_j(t) = r^0(t);$$

2) for any natural  $N \geq 0$

$$\lim_j \int N(\lambda_j - N(j) \lambda_j) dt = 0.$$

With the aid of Theorem 1 and the lemma, the stationarity condition for the trajectory  $x^0, n^0 \mid t$  can be written in the following equivalent form.

There exist  $s_0, \psi(t), r_i(t), m_k(t)$ , a measure  $\nu$ , and a sequence  $\delta_j, r_i^j(t), m_k^j(t)$  such that

$$1) \quad -s_0 F_u + (\psi(t), f_u) + \sum_k m_k(t) g_{ku} - \sum_i r_{iu} = 0;$$

$$2) \quad s_0 \geq 0, \quad \int \sum_k |m_k(t)| dt < +\infty, \quad r_i(t) \in \text{con } N_i(x^0, n^0 \mid t);$$

$$3) \quad -\frac{d\psi}{dt} = f_x^* \psi - s_0 F_x + \sum_k m_k(t) g_{kx} - \sum_i r_{ix}(t) - \frac{d\nu}{dt};$$

4) the measure  $\nu$  is characterized by the following properties:

$$\frac{d\nu}{dt} = \lim_j \text{cl}(c) \left[ \sum_i r_{ix}^j(t) - \sum_k m_k^j(t) g_{kx} \right],$$

$$r_i^j(t) \in \text{con } N_{i\delta_j}(x^0, n^0 | t), \quad \delta_j \rightarrow 0,$$

\* In other words, it is necessary that from  $\Omega_{ij}^*$ ,  $x \geq 0$  it follow that  $x \in \overline{\Omega}_i$ , and from  $\Omega_y^*$ ,  $x \geq 0$  it follow that  $x \in \overline{\Omega}$  (here the bar denotes closure).

$$\lim_j \int \left| \sum_k m_k^j g_{ku} - \sum_i r_{iu}^j \right| dt = 0;$$

5) **the normalization condition**

$$s_0 + \int_0^1 \sum_i |r_i| dt + \int_0^1 \sum_i |\dot{r}_i| dt = 1.$$

Thus the problem is reduced to describing the class of measures  $\nu$ , independent of the existence of the sequence  $\delta_j, m_k^j(t), r_i^j(t)$ . It is obvious that the class of measures  $\nu$  is determined only by the trajectory  $x^0, u^0 | t$  and by the constraints  $g_k(x, u) = 0$ ,  $x, u \in \bar{Q}_i$ , and does not depend on the form of the functional or of the differential relation.

This question is resolved by Theorem 2. Let  $R_0$  be the set of  $t, x, u, \tilde{r}_i, \beta_k$ , where:

- 1)  $x = x^0(t)$ ;
- 2)  $u \in \overline{u^0(t)}$  \*;
- 3)  $\tilde{r}_i \in \text{con } \bar{N}_i(x, u)$ ,  $x, u \in \mathcal{N}_i$ ;  $\tilde{r}_i = 0$ ,  $x, u \notin \mathcal{N}_i$ ,  $\sum_i |\tilde{r}_i| = 1$ ;
- 4)

$$\sum_k \beta_k g_{ku} - \sum_i \tilde{r}_{iu} = 0.$$

We shall call the set  $R_0$  the set of phase points of the trajectory  $x^0, u^0 | t$ .

**Theorem 2.** 1) For any continuous function  $\xi(t)$  ( $\xi \in E^n$ ),  $\int \xi d\nu$  admits the representation

$$\int \xi d\nu = \int_{R_0} \left( \sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{ku}, \xi(t) \right) d\sigma,$$

$$d\sigma \geq 0, \quad \int_{R_0} d\sigma = \sum_i \int |\dot{r}_i^j| dt;$$

2) for any measure  $\sigma \geq 0$  concentrated on  $R_0$ ,

$$\int_{R_0} \sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{kx}, \xi d\sigma$$

admits the representation

$$\int_{R_0} \left( \sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{kx}, \xi \right) d\sigma = c \int \xi dv, \quad \text{where } c > 0,$$

$$\int_{R_0} d\sigma = \sum_i \int |\dot{r}_i^j| dt.$$

Using the representation of the measure  $v$ , we obtain the final form of the stationarity condition.

II. The Euler equation of the problem under investigation has the form

$$0 = -s_0 \int (F_x x' + F_u \bar{u}) dt - c, x'(1) + \sum_k m_k (g_{kx} x' + g_{ku} \bar{u}) - \sum_i \lambda_i (n_x x' + n_u \bar{u}), \quad (\text{B})$$

$$s_0 \geq 0, \quad m_k \in r_{\infty(t)}^*, \quad \lambda_i \in r_{\infty(n,t)}^*,$$

$$\lambda_i \geq 0, \quad \Pi_i \lambda_i = \lambda_i^* * *, \quad s_0 + \sum_i \|\lambda_i\| > 0,$$

$$dx'/dt = f_x x' + f_u \bar{u}, \quad x'(0) = 0.$$

This condition is equivalent to the one found. Let us clarify how they are related. We decompose, in accordance with (3),  $m_k$  and  $\lambda_i$  into the sum of absolutely continuous and singular components. We obtain:

$$-s_0 \int (F_x x' + F_u \bar{u}) dt + c, x'(1) + \sum_k \int m_k(t) (g_{kx} x' + g_{ku} \bar{u}) dt -$$

$$- \sum_i \int (r_{ix} x' + r_{iu} \bar{u}) dt + \sum_k m_k^s (g_{kx} x') - \sum_i \lambda_i^s (n_x x') =$$

\*  $\overline{u^0}(t)$  denotes the closure of the graph of  $u^0(t)$  with respect to the measure (see (3)).

$$** \Pi_i \lambda_i(z(n, t)) = \lim_{\delta \rightarrow 0} \lambda_i(\chi_{\Pi_i^\delta} z(n, t)) \quad (\text{see (3)}).$$

$$= - \sum_k m_k^s (g_{ku}(\bar{u})) + \sum_i \lambda_i^s (n_u \bar{u}), \quad \text{where } \sum_k \int |m_k(t)| dt < +\infty, \quad r_i(t) \in$$

$$\in \text{con } N_i(x^0, u^0 | t), \quad 0 < s_0 + \sum_i \int |r_i| dt + \sum_i \|\lambda_i^s\| < +\infty, \quad m_k^s \in$$

$\in L_\infty^*(t)$ ,  $\lambda_i^s \in L_\infty(n, t)$  are singular components.

Since the left-hand side of the equality is an absolutely continuous functional of  $\bar{u}$ , while the right-hand side is a singular functional of  $\bar{u}$ , they are both equal to zero.

Since  $x'(t)$  is a continuous function, the problem arises of describing the class of measures  $\mu$  admitting the representation:

$$-\int \xi d\mu = \sum_k m_k^s(g_{kx}x') - \sum_i \lambda_i^s(n_x x'), \quad ( )$$

where

$$\sum_k m_k^s(g_{ku}\bar{u}) - \sum_i \lambda_i^s(n_u \bar{u}) = 0, \quad \sum_i \|\lambda_i^s\| > 0, \quad \lambda_i^s \geq 0, \quad \Pi_i \lambda_i^s = \lambda_i^s. \quad ( )$$

This problem is solved as follows. In the space  $L_\infty : z_i(n, t), \bar{u}(t)$  consider the subspace  $\mathcal{L} : g_{ku}\bar{u} = 0, z_i = n_u \bar{u}$  and the cones  $\tilde{\Omega}_i : \lim_{\delta \rightarrow 0} \text{vrai min } z_i(n, t) > 0, n, t \in \Pi_{i\delta}$ . Condition  $(\beta)$  holds if and only if the system  $\mathcal{L}, \dots, \tilde{\Omega}_i, \dots$  has empty intersection. Since the space  $L_\infty = L_1^*$ , we may apply Theorem 1.

As a result we obtain: in order that  $-\int \xi d\mu$  admit the representation  $(\alpha), (\beta)$ , it is necessary and sufficient that:

$$\frac{d\mu}{dt} = \lim_j \text{cl}(c) \left[ \sum_i r_{ix}^j(t) - \sum_k m_k^j(t) g_{kx} \right],$$

where

$$r_i^j(t) \in \text{con } \tilde{N}_{i\delta_j}(x_i^0, u^0 | t), \quad \delta_j \rightarrow 0, \quad c_1 \leq \int \sum_i |r_i^j| dt \leq c_2,$$

$$c_1, c_2 > 0, \quad \lim_j \int \left| \sum_k m_k^j g_{ku} - \sum_i r_{iu}^j \right| dt = 0.$$

Thus we see that the class of measures  $\mu$  coincides with the class of measures  $\nu$  considered in Part I of the note.

In conclusion, we note:

1. The results of [3] are contained in those presented here, for in [3] smooth inequality constraints were considered and there were no constraints of the form  $g_k(x, u) = 0$ .

2. In this note, by means of the concept of the set of phase points of a trajectory, both purely phase and mixed constraints are considered from a unified point of view. Until now, in the literature, mixed constraints have been subjected to a regularity requirement, i.e., the requirement that there be no phase points generated by the mixed constraints.

Institute of Chemical Physics  
Academy of Sciences of the USSR  
Moscow

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*Note: Figure translations are in progress. See original paper for figures.*

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