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Abstract

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MATHEMATICS

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ELLIPTIC PSEUDODIFFERENTIAL OPERATORS ON A CLOSED MANIFOLD DEGENERATING ON A SUBMANIFOLD

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Let on a smooth compact manifold M of dimension n there be given a pseudodifferential operator $P(x, D)$ of order $m > 0$. It is assumed that $P(x, D)$ satisfies the ellipticity condition for $x \in M \setminus \Gamma$, where Γ is a smooth submanifold of dimension $n - 1$. Let a covering $\{U_i\}$ of some neighborhood of the submanifold Γ be fixed, and in each neighborhood U_i let a local coordinate system (l.c.s.) $(x_1, \dots, x_n)$ be so defined that Γ is given by the equation $x_n = 0$, and, for any intersecting U_i and U_j , the passage from the l.c.s. $(x_1^i, \dots, x_n^i)$ in U_i to the l.c.s. $(x_1^j, \dots, x_n^j)$ in U_j is effected by means of a transformation of the form

$$x_1^j = \varphi_1(x_1^i, \dots, x_{n-1}^i), \dots, \quad x_{n-1}^j = \varphi_{n-1}(x_1^i, \dots, x_{n-1}^i),$$

$$x_n^j = \varphi_n(x_1^i, \dots, x_n^i).$$

We note that such transformations preserve the coordinate lines $x' = \text{const}$, where $x' = (x_1, \dots, x_{n-1})$. Denote by ω the manifold $x_n = 0$, $\xi_n = 0$, $\xi' \neq 0$ in the cotangent bundle of the manifold M . Let, in the l.c.s. corresponding to U_i , the full symbol $p(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi)$ of the operator $P(x, D)$ possess the properties:

1. There is given a $\delta > 0$ such that $m\delta$ is an integer and $p_0(x, \xi) = 0$ on ω , $p_{0\beta_n}^{\alpha_n}(x, \xi) = 0$ on ω for $\beta_n < (m - \alpha_n)\delta$, $p_{j\beta_n}^{\alpha_n}(x, \xi) = 0$ on ω for $\beta_n < (m - j - \alpha_n)\delta - j$, where

$$p_{j\beta_n}^{\alpha_n}(x, \xi) = \frac{\partial^{\alpha_n + \beta_n}}{\partial x_n^{\beta_n} \partial \xi_n^{\alpha_n}} p_j(x, \xi) *.$$

2. $p_0(x, \xi) \neq 0$ outside ω and

$$\frac{\partial^m}{\partial \xi_n^m} p_0(x, \xi) \neq 0$$

on ω .

3. For $x_n \neq 0$ and $\xi \neq 0$

$$L_0^0(x', x_n, \xi) = \sum_{\alpha_n=0}^m \frac{1}{\alpha_n! \beta_n!} x_n^{\beta_n} p_{0\beta_n}^{\alpha_n}(x', 0, \xi', 0) \xi_n^{\alpha_n} \neq 0, \quad \beta_n = (m - \alpha_n)\delta, \quad (1)$$

where the summation is carried out only over those indices for which $\beta_n = (m - \alpha_n)\delta$ is an integer.

An example of a symbol with the indicated properties is $p_0(x, \xi) = i\xi_n + ax_n^r |\xi|$, where $\operatorname{Re} a \neq 0$, r is an integer. Operators with such symbols arise in the study of the oblique derivative problem for the Laplace equation. It is known that such operators may have an infinite-dimensional kernel (cokernel).

* Here $p_j(x, \xi)$ are functions of class C^∞ for $\xi \neq 0$ and homogeneous in ξ of order $m - j$. For the definition of pseudodifferential operators see (1).

In the present article the operator $P(x, D)$ is studied in natural functional spaces. It follows from the theorems proved below that, generally speaking, it has an infinite-dimensional kernel and cokernel. However, if additional boundary and coboundary conditions are prescribed on Γ , then the corresponding problem becomes Noetherian (i.e., has finite-dimensional kernel and cokernel). For such problems a right and a left regularizer are constructed below. We note that an analogous question in the case of differential operators degenerating on the boundary of a domain was considered in (2).

Lemma. If the complete symbol $p(x, \xi)$ of the operator $P(x, D)$ in each neighborhood U_i satisfies conditions 1–3, then $p_0(x, \xi)$ can be represented in the form

$$p_0(x, \xi) = q_{m0}^0(x, \xi) \xi_n^m + \sum_{\alpha_n=0}^{m-1} \sum_{m\delta \geq \beta_n \geq (m-\alpha_n)\delta} x_n^{\beta_n} q_{\alpha_n \beta_n}^0(x, \xi) \xi_n^{\alpha_n}, \quad (2)$$

where $q_{\alpha_n \beta_n}^0(x, \xi)$ are homogeneous functions in ξ of orders $m - \alpha_n$ of class C^∞ for $\xi \neq 0$, and β_n are integers. The terms $p_j(x, \xi)$ of the complete symbol of the operator $P(x, D)$ admit an analogous expansion

$$p_j(x, \xi) = q_{j0}^j(x, \xi) \xi_n^j + \sum_{\alpha_n=0}^{m-j-1} \sum_{m\delta \geq \beta_n \geq (m-j-\alpha_n)\delta-j} x_n^{\beta_n} q_{\alpha_n \beta_n}^j(x, \xi) \xi_n^{\alpha_n}, \quad (3)$$

where $q_{\alpha_n \beta_n}^j(x, \xi)$ are homogeneous functions in ξ of order $m - j - \alpha_n$ of class C^∞ for $\xi \neq 0$. For sufficiently small x_n the symbol $q_{m0}^0(x, \xi)$ is elliptic, i.e. for $\xi \neq 0$, $\xi \in R^n$, $q_{m0}^0(x, \xi) \neq 0$.

As in (2), the functional space $H_{(m,\delta)}(M, \Gamma)$ is introduced, with norm $\|\cdot\|_{m,\delta}$, equivalent to the usual norm of the space $H_m(M)$ outside a neighborhood of Γ , and for $u(x)$ with support in a neighborhood of Γ

$$\|u\|_{m,\delta} = \sum_i \sum_{\alpha_n=0}^m \|x_n^{(m-\alpha_n)\delta} (\Lambda')^{m-\alpha_n} D_n^{\alpha_n} \varphi_i u\| + \|u\|,$$

where Λ' is the operator with symbol $1 + |\xi'|$; $\|\cdot\|$ is the norm in $L_2(M)$; φ_i is a partition of unity in a neighborhood of Γ , subordinate to the covering U_i . We note that $H_{(m,\delta)}(M, \Gamma)$ is embedded in $H_{m/(1+\delta)}(M)$.

Suppose that the characteristic equation

$$L_0^0(x', x_n, \xi', \zeta) = 0, \quad (4)$$

for $x_n > 0$ and $\xi' \neq 0$ has μ roots with $\text{Im } \zeta > 0$, and for $x_n < 0$ has ν roots with $\text{Im } \zeta < 0$. Consider the following problem on M :

$$P(x, D)u + \sum_{i=1}^k G_i \rho_i(x') \otimes \delta(\Gamma) = f(x), \quad (5)$$

$$\gamma B_j u + \sum_{i=1}^k E_{ji} \rho_i(x') = g_j(x'), \quad 1 \leq j \leq l, \quad (6)$$

where γ is the operator of restriction of functions to Γ ; E_{ji} are pseudodifferential operators on Γ (with homogeneous symbols $e_{ji}(x', \xi')$); B_j and G_i are pseudodifferential operators on M , and in a neighborhood of Γ the symbols $b_j(x, \xi)$ for B_j are quasihomogeneous of order m_j in ξ , i.e. $b_j(x, \lambda^{1+\delta} \xi', \lambda \xi_n) = \lambda^{m_j} b_j(x, \xi)$, $\lambda > 0$, while the symbols $g_i(x, \xi)$ for G_i are quasihomogeneous in ξ of order σ_i . For nondegenerate elliptic operators such problems were considered in (3, 4).

The number k of coboundary operators in (5) and the number l of boundary operators in (6) must be related to the numbers μ and ν of roots of equation (4) by the formula

$$\mu + \nu - m = l - k. \quad (7)$$

It is assumed that $\sigma_i < -1/2$, $m_j < m - 1/2$, and the order of E_{ji} is equal to $t_i - s_j$, where $s_j = (m - m_j - 1/2)/(1 + \delta)$, $t_i = (\sigma_i + 1/2)/(1 + \delta)$.

Theorem 1. The operator $\mathfrak{A}$:

$$(u, \rho_1, \dots, \rho_k) \xrightarrow{\mathfrak{A}} (f, g_1, \dots, g_l),$$

defined by (5), (6), gives a continuous mapping

$$\begin{aligned} \mathcal{H}_1 &\equiv H_{(m,\delta)}(M, \Gamma) \times H_{t_1}(\Gamma) \times \dots \times H_{t_k}(\Gamma) \xrightarrow{\mathfrak{A}} \\ &\xrightarrow{\mathfrak{A}} L_2(M) \times H_{s_1}(\Gamma) \times \dots \times H_{s_l}(\Gamma) \equiv \mathcal{H}_2. \end{aligned}$$

In order to formulate the conditions for normal solvability of problem (5), (6), consider on the line $\mathbb{R}^1$ the problem

$$\begin{aligned} \frac{1}{q_{m0}^0(x', 0, \xi', 0)} &\left(L_0^0(x', x_n, \xi', D_n)v(x_n) + \sum_{\tau > j \geq 1} L_j^0(x', x_n, \xi', D_n)v(x_n) \right. \\ &\left. + \sum_{i=1}^k \tilde{g}_i(x', 0, \xi', x_n)\rho_i = f_1(x_n), \right. \end{aligned} \quad (8)$$

$$\left(\tilde{b}_j(x', 0, \xi', x_n), v(x_n) \right) + \sum_{i=1}^k e_{ji}(x', \xi')\rho_i = \psi_j, \quad (9)$$

where

$$\tau = \left[\frac{m\delta}{1+\delta} \right],$$

$\tilde{g}_i(x', 0, \xi', x_n)$ and $\tilde{b}_j(x', 0, \xi', x_n)$ are the inverse Fourier transforms of $g_i(x', 0, \xi)$ and $b_j(x', 0, \xi)$ with respect to ξ_n ; $L_j^0(x', x_n, \xi)$ is given by a formula analogous to (1), in which p_0 is replaced by p_j , $\beta_n = (m - \alpha_n)\delta$ is replaced by $\beta_n = (m - j - \alpha_n)\delta - j$, and the summation over α_n is carried out up to $\alpha_n = m - j$. The following is assumed to hold.

Condition $Z_{\xi'}$. The problem (8), (9) with zero right-hand sides f_1 and ψ_j , for any $\xi' \neq 0$, $\xi' \in \mathbb{R}^{n-1}$, has only the trivial solution in the class of functions with finite norm

$$\|v(x_n)\|_{m,\delta}^2 = \sum_{j=0}^m \int (1 + |x_n|)^{2(m-j)\delta} |D_n^j v(x_n)|^2 dx_n. \quad (10)$$

Theorem 2. If conditions 1-3 are satisfied and at each point $x' \in \Gamma$ condition $Z_{\xi'}$ is satisfied, then the operator $\mathfrak{A}$ is Noetherian.

For the proof of the theorem it is sufficient to construct a regularizer, i.e. such a continuous operator R from $\mathcal{H}_2$ to $\mathcal{H}_1$ that $\mathfrak{A}R = I + T_1$, $R\mathfrak{A} = I + T_2$, where I is the identity operator and T_1 and T_2 are completely continuous. Since the operator $P(x, D)$ is elliptic outside Γ , by means of a partition of unity the

construction of R reduces to the construction of a regularizer of the operator $\mathfrak{A}$ in a neighborhood of a point $x'_0 \in \Gamma$. Multiplying equality (5) by the operator with symbol

$$\frac{1}{q_{m0}^0(x, \xi)},$$

we obtain the equation

$$P_1(x, D)u + \sum_{i=1}^k H_i \rho_i(x') \otimes \delta(x_n) = f_1(x). \quad (11)$$

It follows from the composition formula that the operator $P_1(x, D)$ also satisfies conditions 1-3, and for it $q_{m0}^0(x, \xi)$ in the expansion (2) is equal to 1. Next, the operation of freezing the coefficients of the operators B_j , E_{ji} is performed, i.e. they are replaced by operators with symbols $b_j(x'_0, 0, \xi)$, $e_{ji}(x'_0, \xi')$. The freezing operation for $P_1(x, D)$ is somewhat more complicated. The operator $P_1(x, D)$ is replaced by an operator with symbol in which $q_{\alpha\beta_n}^j(x, \xi)$ in (2) and (3) are replaced by $q_{\alpha\beta_n}^j(x'_0, 0, \xi', 0)$, and the terms with

$$\beta_n > (m - j - \alpha_n)\delta - j$$

are discarded. The operator H_i is replaced by an operator with symbol

$$\frac{g_i(x', 0, \xi)}{q_{m0}^0(x', 0, \xi', 0)}.$$

After the Fourier transform with respect to x' , we arrive at problem (8), (9), where f_1 , ψ_j , ρ_i , and v depend on ξ' .

As shown in (2), for $\xi' \neq 0$ on the half-axis $x^n > 0$, equation (8), whose boundary terms have been moved to the right-hand side, has, for arbitrary ρ_i and $f_1(x_n) \in L_2(R_+^1)$, a particular solution $v(x_n)$ with finite norm (10), where the integral is taken over the half-axis; and the homogeneous equation has a μ -parameter family of solutions with the same finite norm. Similarly, for $x_n < 0$ we obtain a ν -parameter family of solutions. Writing out the matching conditions up to order $m-1$ for these two families at $x_n = 0$, and using the boundary conditions, we obtain $m+l$ equations for the arbitrary constants and the unknown densities ρ_i . In view of condition (7), this is a square system of linear equations, which is uniquely solvable, since condition $Z_{\xi'}$ is satisfied. On the basis of this solution one constructs a regularizer R_0 for problem (8), (9), analogously to how this was done in (2).

With the aid of R_0 , a regularizer for problem (11), (6) is constructed in the usual way. Here the essential point is that the symbol

$$x_n^{\beta_n} [q_{\alpha_n\beta_n}^j(x, \xi) - q_{\alpha_n\beta_n}^j(x, \xi', 0)] \xi_n^{\alpha_n}$$

in a sufficiently small neighborhood of the point x'_0 corresponds either to a completely continuous operator or to an operator with small norm. To prove the latter assertion, note that if $q(x, \xi)$ is a homogeneous function in ξ of order $r > 0$, of class C^∞ for $\xi \neq 0$, then

$$q(x, \xi) - q(x, \xi', 0) = Q(x, \xi) \xi_n^r + \sum_{\alpha_n=1}^{r-1} \frac{1}{\alpha_n!} q^{\alpha_n}(x, \xi', 0) \xi_n^{\alpha_n}, \quad (12)$$

where $Q(x, \xi)$, for $\xi \neq 0$, is continuous together with all derivatives with respect to the variables x , and the order of homogeneity of the function $Q(x, \xi)$ in ξ is equal to zero. Therefore $Q(x, \xi)$ serves as the symbol of a bounded operator in $L_2(M)$. Applying expansion (12) to the functions $q^j_{\alpha_n \beta_n}(x, \xi)$ and passing to the corresponding operators, we obtain that our assertion is a consequence of the embedding theorems established in ⁽²⁾. Theorem 1 is proved by analogous arguments.

Let us note that if the right-hand sides $f(x)$ and $g_j(x')$ have additional derivatives in the tangential directions, then the solutions $u(x)$ and $\rho_i(x')$ also have additional smoothness with respect to the tangential variables (the proof is analogous to ⁽²⁾). If $f \in C^\infty(M)$, $g_j \in C^\infty(\Gamma)$, then in some cases it follows from this that $u \in C^\infty(M)$; for example, this is true if there are no boundary operators. By analogous methods one proves

Theorem 3. *Suppose that conditions 1-3 are satisfied. If equation (8) for $\xi' \neq 0$, with $f = 0$ and $\rho_i = 0$, has only the trivial solution with finite norm (10), then the operator $P(x, D)$ is hypoelliptic.*

We note that, as was proved in ⁽²⁾, solutions of equation (8) with $f = 0$ and $\rho_i = 0$ that have finite norm (10) decay at infinity, together with all derivatives, faster than any power.

Remark 1. Above we considered the case in which the symbol has, on Γ , the same order of degeneracy with respect to all variables $\xi_1, \dots, \xi_{n-1}$. Analogously to ⁽²⁾, one may study the case in which the orders of degeneracy with respect to the different variables $\xi_1, \dots, \xi_{n-1}$ are different.

Remark 2. The methods set forth in the present note are also applicable to the study of elliptic systems of pseudodifferential operators degenerating on Γ .

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References

1. J. J. Kohn, L. Nirenberg, *Comm. Pure Appl. Math.*, **18**, No. 1/2, 269 (1965).

2. M. I. Vishik, V. V. Grushin, *Matem. sborn.*, **79** (121), No. 1, 3 (1969).
3. M. I. Vishik, G. I. Eskin, *Matem. sborn.*, **69** (111), No. 1, 65 (1966).
4. B. Yu. Sternin, *DAN*, **172**, No. 1, 44 (1967).

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