

The Sample Mean as an Estimate of a Location Parameter under Certain Losses Different from Quadratic

A. A. Zinger, A. M. Kagan, L. B. Klebanov

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.68288>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.281

Mathematics

A. A. Zinger, A. M. Kagan, L. B. Klebanov

The Sample Mean as an Estimate of a Location Parameter under Certain Losses Different from Quadratic

(Presented by Academician Yu. V. Linnik on 15 III 1969)

1. We consider the problem of estimating the location parameter $\theta \in R^1$ from the data of a repeated sample $(x_1, \dots, x_n)$ from a population with distribution function (d.f.) $F(x - \theta)$. If $\tilde{\theta} = \tilde{\theta}(x_1, \dots, x_n)$ is proposed as an estimate of θ , then the losses incurred thereby are specified by a loss function (loss) $r(\tilde{\theta}; \theta)$, whose mathematical expectation $R(\tilde{\theta}; \theta) = E_{\theta} r(\tilde{\theta}; \theta)$ is called the risk of the estimate $\tilde{\theta}$. For a given risk $R(\tilde{\theta}; \theta)$, the notion of admissibility of an estimate is introduced in the natural way.

If the loss is quadratic, $r(\tilde{\theta}; \theta) = (\tilde{\theta} - \theta)^2$, then for $n \geq 3$ the admissibility of the sample mean $\bar{x} = (x_1 + \dots + x_n)/n$ as an estimate of the parameter θ is equivalent to the normality of $F(x)$ ⁽¹⁾ (see also ⁽²⁾), which also contains some other results clarifying the role of $\bar{x}$). Here analogous results will be formulated concerning losses different from quadratic.

2. Estimation under Laplace loss.

Theorem. Let $(x_1, \dots, x_n)$ be a repeated sample of size $n \geq G$ from a population with d.f. $F(x - \theta)$, where $F(x)$ is single-peaked and has a continuously differentiable density $f(x)$ (single-peakedness means the existence of an x_0 such that $f'(x) \geq 0$ for $x \leq x_0$ and $f'(x) \leq 0$ for $x \geq x_0$). Then the admissibility of the sample mean $\bar{x}$ as an estimate of the parameter $\theta \in R^1$ under Laplace loss $r(\tilde{\theta}; \theta) = |\tilde{\theta} - \theta|$ is an exclusive property of the normal law.

The proof of this theorem is based on combining certain considerations of Pitman ⁽³⁾ with those used in the characterization of distributions by properties of tubular statistics ^(4,5,7). From the admissibility of $\bar{x}$ under Laplace loss one obtains the relation for $f(x)$

$$\int_{-\infty}^0 \prod_1^n f(u + x_i - \bar{x}) du = \int_0^{+\infty} \prod_1^n f(u + x_i - \bar{x}) du, \quad (1)$$

valid for almost all, with respect to the measure $dF(x_1) \cdots dF(x_n)$, values $(x_1, \dots, x_n)$. It is shown that, under the conditions of the theorem, the only probabilistic solution of (1) is the density of the normal law.

The admissibility of the sample mean $\bar{x}$ as an estimate of the population mean from a sample $(x_1, \dots, x_n)$ from a population with normal d.f. $F(x - \theta)$ under Laplace loss follows, for example, from the theorem of Fox and Rubin ⁽⁶⁾.

3. Let us now consider the loss function

$$r(\tilde{\theta}; \theta) = \begin{cases} -\alpha(\tilde{\theta} - \theta), & \tilde{\theta} \leq \theta, \\ \beta(\tilde{\theta} - \theta), & \tilde{\theta} \geq \theta, \end{cases} \quad (2)$$

where $\alpha > 0$, $\beta > 0$ are constants, which generalize the Laplace loss function. Wishing to carry over to the loss (2) the result of the preceding item, one must only bear in mind that for Gaussian quantities $x_1, \dots$

$\dots, x_n$ with $E_\theta x_i = 0$, the sample mean $\bar{x}$ is inadmissible as an estimate of θ under loss (2). It is easy to see that $\min E_\theta r(\bar{x} - c; \theta)$ is attained at c' , defined by the condition $F(c') = \beta/(\alpha + \beta)$.

Under the conditions of Theorem 2, admissibility of the estimate $\bar{x} - c'$ of the location parameter θ under loss (2) is a characteristic property of the normal law $N(0, \sigma^2)$ for some $\sigma^2 \geq 0$.

4. Our considerations are also applicable to loss functions

$$r(\bar{\theta}; \theta) = |\bar{\theta} - \theta|^{2m+1}, \quad (3)$$

where m is a natural number. If the sample mean $\bar{x}$ is admissible as an estimate of θ under loss (3), then the density $f(x)$ satisfies, for almost all values $(x_1, \dots, x_n)$ with respect to the measure $dF(x_1) \cdots dF(x_n)$, the relation

$$\int_{-\infty}^0 |u|^{2m} \prod_1^n f(u + x_i - \bar{x}) du = \int_0^{+\infty} |u|^{2m} \prod_1^n f(u + x_i - \bar{x}) du. \quad (4)$$

It follows easily from (4) that if $f(x) = 0$ for $x \geq x_0$ (or for $x \leq x_0$), then $f(x) = 0$ almost surely with respect to Lebesgue measure. Therefore a continuous unimodal density satisfying (4) is nowhere equal to zero.

In the class of densities $f(x)$ satisfying the following conditions: $f(x)$ is continuously differentiable $2m + 1$ times; the functions $\varphi^{(2k)}(x) = \frac{d^{2k}}{dx^{2k}} \log f(x)$, $k = 1, \dots, m$, preserve their sign; there exists an x_0 such that $\varphi^{(k+1)}(x) \geq 0$ for $x \leq x_0$ and $\varphi^{(2k+1)}(x) \leq 0$ for $x \geq x_0$, $k = 0, 1, \dots, m$ (where the choice of the inequality signs may be different for each k), for $n \geq 6$ the only probabilistic solution of equation (4) is the density of the normal law.

The conditions imposed here a priori on $f(x)$ can apparently be substantially weakened. In any case, in the theorem the requirement of unimodality of $f(x)$ can be replaced by unimodality of the function $f(x)e^{ax}$ for some a .

The authors are indebted to Yu. V. Linnik, who pointed out the connection between the characterization of distributions, the admissibility of certain estimates, and the theory of tubular statistics.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
of the Academy of Sciences of the USSR

Leningrad State University
named after A. A. Zhdanov

Received
7 III 1968

REFERENCES

1. A. M. Kagan, Yu. V. Linnik, C. R. Rao, *Sankhyā*, **A27**, 2-3-4 (1965).
2. A. M. Kagan, *Trudy Matem. Inst. im. V. A. Steklova AN SSSR*, **104** (1968).
3. E. G. Pitman, *Biometrika*, **30**, III-IV (1938).
4. D. V. Anosov, *Vestn. Leningrad. Univ.*, **7** (1964).
5. A. A. Zinger, *ibid.*, **1** (1956).
6. M. Fox, H. Rubin, *Ann. Math. Stat.*, **35**, 3 (1964).
7. A. A. Zinger, Yu. V. Linnik, *Trudy Matem. Inst. im. V. A. Steklova AN SSSR*, **111** (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.