

# NEW CONDITIONS FOR COMPLETE SOLVABILITY

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## NEW CONDITIONS FOR COMPLETE SOLVABILITY

## AND QUESTIONS OF REDUCIBILITY

*(Presented by Academician S. L. Sobolev on 18 X 1968)*

Let  $E_x$  and  $E_y$  be finite-dimensional Banach spaces, the first of them real and the second complex. Denote by  $L(E_x; E_y)$  the Banach space of all linear operators acting from  $E_x$  into  $E_y$ , and let  $L(E_x; E'_y) = L(E_x; L(E_y; E_y))$ .

In this paper we study linear multidimensional differential equations of the form

$$Y'(x)h = A(x)hY(x) \quad (\forall h \in E_x), \quad (1)$$

$$y'(x)h = A(x)hy(x) + f(x)h \quad (\forall h \in E_x), \quad (2)$$

where  $A(x)$  and  $f(x)$  are operator functions defined on a convex domain  $G_x \subset E_x$  and taking values in the spaces  $L(E_x; E'_y)$  and  $L(E_x; E_y)$ , respectively. In the first equation the unknown is an operator function  $Y(x)$  with values in  $L(E_y; E_y)$ , while in the second equation it is a vector function  $y(x)$  with values in the space  $E_y$ . Recall that a multidimensional differential equation is called *completely solvable* if every Cauchy problem is uniquely solvable.

The paper gives new conditions (necessary and sufficient) for complete solvability of multidimensional differential equations and studies the method of successive approximations; for the case of an almost periodic right-hand side, conditions for complete solvability are formulated in terms of Fourier coefficients and frequencies; with the aid of an estimate for the resolvent of a permutable operator, conditions are indicated for the reducibility of an equation with almost periodic coefficients.

Let  $\varphi(x)$  be a continuous operator function defined on the domain  $G_x$  and taking values in  $L(E_x; E_y)$ , and let  $\partial s_\varepsilon$  be the boundary of the triangle  $s_\varepsilon = s(x; \varepsilon h, \varepsilon k)$ , whose positive orientation is specified by the order of the vertices  $x, x + \varepsilon h, x + \varepsilon k$ . Put

$$\operatorname{rot} \varphi(x; h, k) = \lim_{\varepsilon \rightarrow +0} \frac{1}{\varepsilon^2} \int_{\partial s_\varepsilon} \varphi(\xi) d\xi, \quad (3)$$

if the limit exists. In the case where  $\operatorname{rot} \varphi(x; h, k)$  is a bilinear operator in  $h, k$ , we write  $\operatorname{rot} \varphi(x)hk$ . Note that if  $\varphi(x)$  is differentiable, then  $\operatorname{rot} \varphi(x)hk = \Lambda_{hk} \varphi'(x)hk$  ( $\Lambda_{hk} Bhk = (Bhk - Bkh)/2$ ).

**1.** Of great interest in the study of multidimensional differential equations is the question of conditions for complete solvability. If the right-hand side is continuously differentiable, then the answer is given by Frobenius' theorem (see, for example, <sup>(1)</sup>, p. 356). In the monograph <sup>(2)</sup>, R. Nevanlinna gives the condition for complete solvability of the operator equation (1) in the form

$$U(x; h, k) = 1 + o(\delta^2), \quad (4)$$

where  $U(x; h, k)$  is the multiplicative integral of  $A(x)$ , computed over  $\partial s = \partial s(x; h, k)$ , and  $\delta$  is the diameter of the triangle  $s$ . Developing this assertion, we arrive at the following theorem.

**Theorem 1.** Let  $A(x)$  and  $f(x)$  be continuous. Then complete solvability of equation (2) holds if and only if the conditions ( $\forall h, k \in E_x$ )

$$\operatorname{rot} A(x)hk = \Lambda_{hk} A(x)hA(x)k, \quad (5)$$

$$\operatorname{rot} f(x)hk = \Lambda_{hk} A(x)hf(x)k. \quad (6)$$

The proof is based on the formulas

$$U(x; h, k) = 1 + \int A(\xi) d\xi - \Lambda_{hk} A(x)hA(x)k + o(\delta^2), \quad (7)$$

$$\int_{\partial s} f(\xi) d\xi = \Lambda_{hk} A(x)hf(x)k + o(\delta^2), \quad (8)$$

which hold for any compact set  $K \subset G_x$ , the above-cited assertion of R. Nevanlinna and one result of V. Shapiro <sup>(3)</sup>.

We shall seek a solution of equation (2) satisfying the initial condition  $y(\xi) = \eta$ , by the method of successive approximations

$$y_0(x) \equiv \eta$$

$$y'_p(x)h = A(x)hy_{p-1}(x) + f(x)h, \quad y_p(\xi) = \eta \quad (9)$$

$$(p = 1, 2, \dots).$$

**Theorem 2.** The method of successive approximations (9) is realizable for arbitrary initial conditions if and only if  $(\forall h, k \in E_x)$

$$\operatorname{rot} A(x) = 0, \quad \Lambda_{hk} A(x) h A(x) k = 0; \quad (10)$$

$$\operatorname{rot} f(x) = 0, \quad \Lambda_{hk} A(x) h f(x) k = 0. \quad (11)$$

Let  $U(A; f)$  denote the totality of all continuous vector functions  $u(x)$  for which the curvilinear integral of  $\varphi(x)h = A(x)hu(x) + f(x)h$  is path-independent. It can be shown that  $U(A; f)$  contains all constants and is mapped into itself by the integral operator

$$I_\eta u(x) = \eta + \int_\xi^x A(\zeta) d\zeta u(\zeta) + \int_\xi^x f(\zeta) d\zeta \quad (12)$$

for any  $\eta \in E_y$ , if and only if conditions (10) and (11) are satisfied.

2. Let us now consider equation (2) with almost periodic operator functions  $A(x)$  and  $f(x)$ , and let

$$A(x) \sim \sum A_\lambda e^{i\lambda x}, \quad f(x) \sim \sum f_\lambda e^{i\lambda x}; \quad (13)$$

in this case  $G_x = E_x$ ,  $\lambda \in E_x^*$  (see (7)).

**Theorem 3.** Complete solvability for equation (2) with almost periodic operator functions  $A(x)$  and  $f(x)$  holds if and only if  $(\forall h, k \in E_x)$

$$\Lambda_{hk} i\lambda h A_\lambda k = \Lambda_{hk} \left( \sum_{\mu+\nu=\lambda} A_\mu h A_\nu k \right), \quad (14)$$

$$\Lambda_{hk} i\lambda h f_\lambda k = \Lambda_{hk} \left( \sum_{\mu+\nu=\lambda} A_\mu h f_\nu k \right). \quad (15)$$

We emphasize that no smoothness of  $A(x)$  and  $f(x)$  is assumed in the theorem. The proof of Theorem 3 is based on the use of Theorem 1 and the following assertion (the theorem on the rotor).

**Theorem 4.** Let  $f(x)$  be an almost periodic function, and suppose that one can specify an almost periodic function  $R(x)$  with values

in  $L_2(E_x; E_y)$ , such that for any  $h, k \in E_x$

$$R(x)hR \sim \sum \Lambda_{hk}(i\lambda hf, \lambda k)e^{i\lambda x}. \quad (16)$$

Then

$$\int_{\partial s} f(\xi) d\xi = \iint_s R(\xi) d\xi_1 d\xi_2. \quad (17)$$

In formula (17), on the right there is the affine integral (see <sup>(2)</sup>);  $L_2(E_x; E_y)$  is the space of bilinear operators defined on  $E_x \oplus E_x$  and with values in  $E_y$ . Theorem 4 is most simply proved with the aid of Bochner-Fejér polynomials (see, for example, <sup>(4)</sup>), whose theory is not difficult to carry over to vector-valued (and operator-valued) functions.

3. An operator  $A \in L[E_x; E_y]$  is called **permutable** if  $\Lambda_{hk}AhAk = 0$ . A linear functional  $\lambda$  is called an **eigenfunctional** of the operator  $A$  if  $Ah\xi = (\lambda h)\xi$  ( $0 \neq \xi \in E_y$ ) (see <sup>(5)</sup>). The totality  $\sigma(A)$  of all eigenfunctionals is called the **spectrum** of the operator  $A$ . The least natural number  $p$  for which  $(\lambda h - Ah)^{p+1}y = 0$  implies  $(\lambda h - Ah)^py = 0$  is called the **index**  $n(\lambda)$  of the functional  $\lambda$ .

It can be shown that  $\lambda \in \sigma(A)$  if and only if  $\det(\lambda h - Ah) = 0$ . In expanded form this equation has the form

$$(\lambda h)^n - p_1(h)(\lambda h)^{n-1} + \dots + (-1)^n p_n(h) = 0, \quad (18)$$

where  $p_j(h)$  is a homogeneous functional of degree  $j$ . In order to eliminate  $h$  from the written equation, call the **product**  $f\varphi$  of symmetric functionals  $f$  and  $\varphi$  the symmetric functional uniquely determined by the formula  $(f\varphi)h^{p+q} = (fh^p)(\varphi h^q)$  ( $f$  and  $\varphi$  are assumed to be  $p$ - and, respectively,  $q$ -linear functionals). Let  $p_j$  be the symmetric  $j$ -linear functional uniquely determined from the relation  $p_j h^j = p_j(h)$  (as above, we use here the result from <sup>(6)</sup>, p. 769). Then equation (18) takes the form

$$\mathfrak{D}(\lambda) \stackrel{\text{def}}{=} \lambda^n - p_1 \lambda^{n-1} + \dots + (-1)^n p_n = 0. \quad (19)$$

Let us note that for a permutable operator the formula

$$\mathfrak{D}(\lambda) = \prod_{\mu \in \sigma(A)} (\lambda - \mu)^{n(\mu)} \quad (20)$$

is valid.

If  $\lambda \notin \sigma(A)$ , then the equation  $(\lambda h - Ah)y = fh$  is uniquely solvable if and only if  $\Lambda_{hk}(\lambda h - Ah)fk = 0$  ( $f \in L(E_x; E_y)$ ). The inverse operator  $R(\lambda; A)$  will be called the **resolvent** of the permutable operator  $A$ .

**Theorem 5.** Let  $A$  be a permutable operator and  $\lambda \notin \sigma(A)$ . Then

$$\|R(\lambda; A)\| \leq c(A) \left( \sum_{j=1}^{n(A)} \frac{1}{[\rho(\lambda)]^j} \right), \quad (21)$$

where  $\rho(\lambda)$  is the distance from  $\lambda$  to  $\sigma(A)$ ;  $n(A)$  is the maximum of the indices of the eigenfunctionals of the operator  $A$ .

Let us now consider the operator  $\mathfrak{C} \in L\{E_x; E_y\}$ , given by the formula

$$\mathfrak{C}hY = AhY + YBh, \quad (22)$$

where  $L\{E_x; E_y\} = L[E_x; L(E_y; E_y)]$  and  $A, B \in L[E_x; E_y]$ ,  $Y \in L(E_y; E_y)$ .

**Theorem 6.** The operator  $\mathfrak{C}$  is permutable if and only if the operators  $A$  and  $B$  are permutable;  $\sigma(\mathfrak{C}) = \sigma(A) + \sigma(B)$ , and

$$n(\nu) \leq \max_{\lambda+\mu=\nu} [n(\lambda) + n(\mu)] - 1,$$

where  $\lambda \in \sigma(A)$ ,  $\mu \in \sigma(B)$ , and  $\nu \in \sigma(\mathfrak{C})$ .

4. Consider the operator equation, completely solvable for all  $\varepsilon$ ,

$$Y'(x)h = (A + \varepsilon B(x))hY(x), \quad (23)$$

where  $A \in L[E_x; E_y]$ , and the operator function  $B(x)$  is an almost periodic function,  $B(x) \sim \sum B_\mu e^{i\mu x}$ , with  $B_0 = 0$ . The operator  $A$  is permutable, and suppose that its spectrum consists of the functionals  $\lambda^1, \dots, \lambda^p$ . Denote by  $\sigma_B$  the spectrum of the almost periodic function  $B(x)$ , and let  $j\sigma_B$  be the set of all sums of the form  $\mu^1 + \dots + \mu^j$ , where  $\mu^i \in \sigma_B$  ( $i = 1, \dots, j$ ). Suppose that  $\|B(\cdot)\|_* = \sum \|B_\mu\| < \infty$  and

$$\delta_j = \inf \|i\nu - (\lambda^t - \lambda^s)\|, \quad (24)$$

where  $\nu \in j\sigma_B$  and  $t, s = 1, 2, \dots, p$ . Put

$$\Delta_j = \sum_{k=1}^{2n(A)-1} \frac{1}{\delta_j^k}, \quad (25)$$

$$\frac{1}{\varkappa} = c(\mathfrak{C})\|B(\cdot)\|_* \lim_{p \rightarrow \infty} (\Delta_1 \dots \Delta_p)^{1/p}, \quad (26)$$

where  $c(\mathfrak{C})$  is the constant in an estimate of type (21) for the resolvent of the operator  $\mathfrak{C}$ , where  $\mathfrak{C}hY = AhY + YBh$ .

**Theorem 7.** *Under the assumptions stated above, the solution of equation (23) with  $Y(0, \varepsilon) = 1$  is representable in the form*

$$Y(x, \varepsilon) = Z(x, \varepsilon)e^{Ax}, \quad (27)$$

where the series

$$Z(x, \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j Z_j(x) \quad (28)$$

converges uniformly and absolutely for  $|\varepsilon| < \varkappa$ , and each  $Z_j(x)$  is an almost periodic operator function whose spectrum lies in  $j\sigma_B$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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