

# A NEW BOUNDARY-VALUE PROBLEM FOR A DEGENERATING HYPERBOLIC EQUATION

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## Abstract

## Full Text

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MATHEMATICS

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# A NEW BOUNDARY-VALUE PROBLEM FOR A DEGENERATING HYPERBOLIC EQUATION

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As a model degenerating hyperbolic equation of the first kind, of second order, in two independent variables, consider the equation

$$y^m u_{xx} - u_{yy} = 0, \quad m = \text{const} > 0. \quad (1)$$

Let  $\Delta$  be a finite simply connected domain of the plane of the complex variable  $z = x + iy$ , bounded by the characteristics  $AC$  and  $BC$  of equation (1), issuing from the points  $A(0,0)$ ,  $B(1,0)$ , and by the segment  $AB$  of the straight line  $y = 0$ ;  $I$  is the unit interval,  $0 < x < 1$ ;  $z = \theta_0(x)$  and  $z = \theta_1(x)$  are the points of intersection of the characteristics of equation (1) issuing from the point  $z = x \in I$ , with the characteristics  $AC$  and  $BC$ , respectively;

$$D_{0x}^l f(x) \equiv \begin{cases} \frac{1}{\Gamma(-l)} \int_0^x \frac{f(t) dt}{(x-t)^{1+l}}, & l < 0, \\ \frac{d^{n+1}}{dx^{n+1}} D_{0x}^{l-(n+1)} f(x), & l > 0, \end{cases} \quad (2)$$

$$D_{x1}^l f(x) \equiv (-1)^k D_{0y}^l f(1-y), \quad y = 1-x, \quad (3)$$

where  $l$  is any real number;  $n$  is the integer part of  $l \geq n$ ;  $\Gamma(z)$  is the gamma function;  $k = 0$  ( $k = n$ ) for  $l < 0$  ( $l > 0$ ).

Obviously,  $D_{0x}^l$  and  $D_{x1}^l$  are operators of fractional integration of order  $(-l)$  for  $l < 0$  and generalized derivatives (in the sense of Liouville) of order  $l$  for  $l > 0$  <sup>(1)</sup>.

**Problem 1.** Find a function  $u(x, y) \equiv u(z)$  with the following properties:

1.  $u(z) \in C(\overline{\Delta}) \cap C^1(\Delta \cup I)$ ,

$$\int_0^1 u_y(x, 0)[x(1-x)]^{-\varepsilon} dx < \infty.$$

2.  $u(z)$  is a regular solution in the domain  $\Delta$  of equation (1), satisfying the boundary conditions

$$u(x) = \tau(x) \quad \forall x \in \bar{I}, \quad (4)$$

$$\alpha(x)D_{0x}^{1-\varepsilon}u[\theta_0(x)] + \beta(x)D_{x1}^{1-\varepsilon}u[\theta_1(x)] = \gamma(x), \quad (5)$$

$$\forall x \in I, \quad \text{where } 2(m+2)\varepsilon = m, \quad u[\theta_k(x)] = u(\operatorname{Re} \theta_k, \operatorname{Im} \theta_k),$$

$$\tau(x) \in C^1(\bar{I}) \cap C^3(I), \quad (6)$$

$$\alpha(x), \beta(x), \gamma(x) \in C(\bar{I}) \cap C^2(I); \quad (7)$$

$$\alpha(x)(1-x)^\varepsilon + \beta(x)x^\varepsilon \neq 0 \quad \forall x \in \bar{I}. \quad (8)$$

We shall prove that this problem is well posed.

Relying on the unique solvability of the Cauchy problem for equation (1) with data on the line of degeneracy  $y = 0$  (see (2)), it is easy to verify,

that any solution  $u(z)$  of problem 1, if it exists, is representable in the form

$$\begin{aligned} u(z) = & \frac{\Gamma(2\varepsilon)}{\Gamma^2(\varepsilon)} \int_0^1 \tau \left[ x + \frac{2}{m+2} y^{(m+2)/2} (2t-1) \right] [t(1-t)]^{\varepsilon-1} dt + \\ & + \frac{\Gamma(2-2\varepsilon)}{\Gamma^2(1-\varepsilon)} y \int_0^1 \nu \left[ x + \frac{2}{m+2} y^{(m+2)/2} (2t-1) \right] [t(1-t)]^{-\varepsilon} dt, \end{aligned} \quad (9)$$

where  $\nu(x) = u_y(x, 0)$ .

From (9), in view of (2) and (3), after simple transformations we have

$$u[\theta_0(x)] = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)} x^{1-2\varepsilon} D_{0x}^{-\varepsilon} x^{\varepsilon-1} \tau(x) - \left( \frac{m+2}{4} \right)^{1-2\varepsilon} \frac{\Gamma(2-2\varepsilon)}{\Gamma(1-\varepsilon)} D_{0x}^{\varepsilon-1} x^{-\varepsilon} \nu(x), \quad (10)$$

$$u[\theta_1(x)] = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)}(1-x)^{1-2\varepsilon}D_{x1}^{-\varepsilon}(1-x)^{\varepsilon-1}\tau(x) - \left(\frac{m+2}{4}\right)^{1-2\varepsilon} \frac{\Gamma(2-2\varepsilon)}{\Gamma(1-\varepsilon)}D_{x1}^{\varepsilon-1}(1-x)^{-\varepsilon}\nu(x). \quad (11)$$

Substituting (10) and (11) into the boundary condition (5) and taking into account the fact that

$$D_{0x}^{1-\varepsilon}D_{0x}^{\varepsilon-1} = D_{x1}^{1-\varepsilon}D_{x1}^{\varepsilon-1} = D^0,$$

where  $D^0$  is the identity operator, we obtain

$$\begin{aligned} & \left(\frac{m+2}{4}\right)^{1-2\varepsilon} \frac{\Gamma(2-2\varepsilon)}{\Gamma(1-\varepsilon)} [\alpha(x)(1-x)^\varepsilon + \beta(x)x^\varepsilon] \nu(x) = \\ & = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)} [x(1-x)]^\varepsilon [\alpha(x)D_{0x}^{1-\varepsilon}x^{1-2\varepsilon}D_{0x}^{-\varepsilon}x^{\varepsilon-1}\tau(x) + \\ & + \beta(x)D_{x1}^{1-\varepsilon}(1-x)^{1-2\varepsilon}D_{x1}^{-\varepsilon}(1-x)^{\varepsilon-1}\tau(x) - \frac{\Gamma(\varepsilon)}{\Gamma(2\varepsilon)}\gamma(x)]. \end{aligned} \quad (12)$$

Further, taking (2) into account, one can write

$$\begin{aligned} D_{0x}^{1-\varepsilon}x^{1-2\varepsilon}D_{0x}^{-\varepsilon}x^{\varepsilon-1}\tau(x) &= \tau_1(x) = \frac{1}{\Gamma^2(\varepsilon)} \frac{d}{dx} \int_0^x \frac{t^{1-2\varepsilon}dt}{(x-t)^{1-\varepsilon}} \int_0^t \frac{\tau(\xi)d\xi}{[\xi(t-\xi)]^{1-\varepsilon}} \\ &= \frac{1}{\Gamma(2\varepsilon-1)} \frac{d}{dx} \int_0^1 \frac{x^\varepsilon\tau(xt)}{t^{1-\varepsilon}(1-t)^{1-2\varepsilon}} F(2\varepsilon-1, \varepsilon, 2\varepsilon, 1-t) dt, \end{aligned} \quad (13)$$

where  $F$  is the Gauss hypergeometric function.

From the equalities (13), on the basis of condition (6), we conclude that the function  $\tau_1(x) \in C(I \cup B) \cap C^2(I)$ , and at the point  $x = 0$  it may tend to infinity of order not higher than  $1 - \varepsilon$ .

In a completely analogous way we verify that the function

$$D_{x1}^{1-\varepsilon}(1-x)^{1-2\varepsilon}D_{x1}^{-\varepsilon}(1-x)^{\varepsilon-1}\tau(x) \in C(I \cup A) \cap C^2(I),$$

and at the point  $x = 1$  it may tend to infinity of order not higher than  $1 - \varepsilon$ .

Now, if conditions (8) are taken into account, it is easy to see that the function  $\nu(x)$  is uniquely determined from relation (12), and, in view of (7), it is twice

continuously differentiable in the interval  $I$ , possibly tending to infinity of order not higher than  $1 - 2\varepsilon$  at its endpoints.

Consequently, problem 1 is uniquely and unconditionally solvable, and its solution  $u(z)$  is given by Darboux formula (9), where  $\nu(x)$  is determined from (12).

From problem 1, for  $\beta(x) \equiv 0$ ,  $\alpha(x) \equiv 1$ , one obtains the known mixed Darboux problem (3):

$$u(x) = \tau(x), \quad u[\theta_0(x)] = D_{0x}^{\varepsilon-1} \gamma(x) \quad \forall x \in \bar{I}.$$

Suppose that for all  $x \in I$  condition (8) is violated, i.e.  $\beta(x)x^\varepsilon + \alpha(x)(1-x)^\varepsilon \equiv 0$ , and  $\alpha(x) \neq 0$ . Then from (12), taking (5) into account, we shall have

$$x^\varepsilon D_{0x}^{1-\varepsilon} u[\theta_0(x)] - (1-x)^\varepsilon D_{x1}^{1-\varepsilon} u[\theta_1(x)] = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)} [x^\varepsilon D_{0x}^{1-\varepsilon} x^{1-2\varepsilon} D_{0x}^\varepsilon x^{\varepsilon-1} u(x) - (1-x)^\varepsilon D_{x1}^{1-\varepsilon} (1-x)^{1-2\varepsilon} D_{x1}^\varepsilon (1-x)^{\varepsilon-1} u(x)].$$

It is known <sup>(3)</sup> that any solution  $u(z)$ , regular in the domain  $\Delta$ , of equation (1) for  $m = 0$ , i.e. of the wave equation

$$u_{xx} - u_{yy} = 0, \quad (15)$$

which satisfies condition 1 of Problem 1 for  $\varepsilon = 0$ , possesses the following very important property (of the mean value):

$$\frac{d}{dx} u[\theta_0(x)] + \frac{d}{dx} u[\theta_1(x)] = \frac{d}{dx} \left[ u(x) + u\left(\frac{1}{2}, \frac{1}{2}\right) \right], \quad (16)$$

where  $\theta_k(x)$  ( $k = 0, 1$ ) coincide with the function  $\theta_k(x)$  defined above for  $m = 0$ .

It may be considered that formula (14) expresses an analogous property of all regular solutions of equation (1) (satisfying condition 1 of Problem 1).

A direct generalization of Problem 1 is evidently the boundary-value problem in which condition (5) is replaced by a condition that pointwise relates the values of generalized derivatives of order  $l \geq 0$  on  $AC \cup BC$ .

Here we shall restrict ourselves to the investigation of one boundary-value problem of this kind for equation (15) in the domain  $\Delta$ , bounded by the straight lines  $y = x$ ,  $y = 1 - x$ ,  $y = 0$ .

Below, by a solution of equation (15) in the domain  $\Delta$  we mean any function  $u(z)$  representable in the form

$$u(z) = \varphi(x + y) + \psi(x - y), \quad (17)$$

where  $\varphi(x)$  and  $\psi(x) \in C(I)$ .

**Problem 2.** Find a solution  $u(z)$  of equation (15), continuous in  $\bar{\Delta}$  and satisfying the boundary conditions

$$u(x) = \tau(x) \quad \forall x \in \bar{I}, \quad (18)$$

$$u\left(\frac{x}{2}, \frac{x}{2}\right) + \beta(x) u\left(\frac{1+\theta(x)}{2}, \frac{1-\theta(x)}{2}\right) = \gamma(x), \quad \forall x \in \bar{I}, \quad (19)$$

where  $\tau(x)$ ,  $\beta(x)$ ,  $\theta(x)$ , and  $\gamma(x)$  are prescribed functions of class  $C(\bar{I})$ , with  $0 \leq \theta(x) \leq 1$ .

Problem 2 is solvable, and moreover in a unique way, if the following conditions are fulfilled:

1. The function  $y = \theta(x)$  effects a topological mapping of the segment  $\bar{I}$  onto itself, leaving the points  $x = 0, 1$  fixed.
2.  $|\beta(x)| \neq 1 \quad \forall x \in \bar{I}, \quad \gamma(0) = 0. \quad (20)$
3.  $\tau(0) = 0, \quad \gamma(1) = \beta(1)\tau(1). \quad (21)$

Taking (18) and (19) into account, we note that, under the fulfillment of requirements 1 and 3, the condition  $\gamma(0) = 0$  is necessary for the solvability of Problem 2.

Relying on the property (of the mean value) (16) of the solution (17) and on relations (18), (19), (20), and (21), it is not difficult to verify the equivalence of Problem 2 to the two-point boundary-value problem

$$f(0) = 0, \quad f(1) = 0. \quad (22)$$

for the functional equation

$$Af \equiv \beta(x)f[\theta(x)] + \mu(x) = f(x) \quad \forall x \in \bar{I}, \quad (23)$$

where  $f(x) = u(x/2, x/2)$ ,  $\mu(x) = \gamma(x) - \beta(x)t[\theta(x)]$ .

Consider the space  $C_0(\bar{I})$  of functions  $f(x) \in C(\bar{I})$  with  $\|f\| = \max_{\bar{I}} |f(x)|$  and satisfying condition (22), which is a Banach space.

It is easy to see that the operator  $A$  from (23) maps the space  $C_0(\bar{I})$  into itself and, in the case when  $\|\beta\| < 1$  (see (20)), in view of the obvious inequality

$$\|Af_1 + Af_2\| \leq \|\beta\| \|f_1[\theta(x)] - f_2[\theta(x)]\| = \|\beta\| \|f_1 - f_2\|,$$

valid for any  $f_1$  and  $f_2$  from  $C_0(\bar{I})$ , is a contraction operator. Consequently, by Banach's theorem (<sup>4</sup>), equation (23) has a solution, and a unique one; moreover, it can be constructed by the method of successive approximations.

The case  $\|\beta\| > 1$ , by replacing  $x$  by  $\theta^{-1}(x)$  in equation (23), is reduced to the one already considered.

Examples may be given showing that violation of one of the first two conditions imposed on the given functions affects the well-posedness (in the sense of Hadamard) of problem 2 (see (<sup>5</sup>)).

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*Note: Figure translations are in progress. See original paper for figures.*

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