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Abstract

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PHYSICS

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MOTION OF A MEDIUM WITH ULTRARELATIVISTIC VELOCITIES IN THE GENERAL THEORY OF RELATIVITY

(Presented by Academician Ya. B. Zel'dovich, 9 X 1968)

The basic equations describing the motion of a medium in a centrally symmetric field in the general theory of relativity, and the field itself, will be written in the form¹

$$\frac{1}{c^2\theta^2} [Au_t + uu_r] - \frac{\omega^2}{c^2} \left[(\ln v)_r - \frac{Au}{c^2} (\ln v)_t \right] = \frac{1}{2u} [A\lambda_t + u\lambda_r] + \frac{\theta^2 T^0 \sigma_r}{W}; \quad (1)$$

$$-[A(\ln v)_t + u(\ln v)_r] + \frac{1}{\theta^2} \left[u_r + \frac{Auu_t}{c^2} \right] + \frac{2u}{r} = \frac{u}{2} \left[\lambda_r + \frac{Au\lambda_t}{c^2} \right]; \quad (2)$$

$$A\sigma_t + u\sigma_r = 0. \quad (3)$$

Here $A = e^{(\lambda-\nu)/2}$; $u = A dr/dt$; $\theta^2 = 1 - u^2/c^2$; $W = (p + \varepsilon)v$; u is the 3-velocity; p is the pressure; v is the specific volume; $\varepsilon = \rho c^2$ is the energy density; W is the heat content, $\omega^2/c^2 = -(\partial \ln W / \partial \ln v)_\sigma$; ω is the speed of sound; σ is the entropy; T^0 is the temperature. In addition one must know the equation of state of the medium $p = p(\sigma; v)$ and use the identity $\partial(p; v)/\partial(T; \sigma) = 1$.

We shall write the two independent field equations in the form:

$$(re^{-\lambda})_r = 1 - \frac{\chi r^2}{\theta^2} [\varepsilon + pu^2/c^2]; \quad (4)$$

$$A(re^{-\lambda})_t = \frac{\chi ur^2}{\theta^2} [\varepsilon + p]. \quad (5)$$

Let us consider the motion when $u/c = 1 - 2\Delta$, where $\Delta \ll 1$,

$$p = \sigma v^{-k} = (k-1)\varepsilon; \quad \omega^2/c^2 = (k-1); \quad (6)$$

neglecting terms of order Δ^2 , we arrive at the system of equations

$$Ax_\tau + x_r = 0, \quad (7)$$

where $x = \ln[p r^{2k/(2-k)}]$, $\tau = ct$;

$$Ay_\tau + y_r = 0, \quad (8)$$

where $y = \ln[\Delta e^{\lambda} r^{4(k-1)/(2-k)}]$;

$$A\sigma_\tau + \sigma_r = 0; \quad (9)$$

$$\lambda_r = \frac{1}{r} + e^\lambda \left[\frac{\chi r}{4\Delta} (\varepsilon + p) - \left(\frac{1}{r} + \chi r p \right) \right]; \quad (10)$$

$$A\lambda_\tau = -\frac{\chi r^2}{4\Delta} e^\lambda (\varepsilon + p). \quad (11)$$

We introduce a new independent variable m by means of the relation

$$\chi \left(\frac{\partial m}{\partial r} \right)_t = \frac{\chi r^2}{\theta^2} \left(\varepsilon + \frac{p u^2}{c^2} \right) = 1 - (r e^{-\lambda})_r,$$

whence

$$m = \int_0^r \frac{r^2}{\theta^2} \left(\varepsilon + p \frac{u^2}{c^2} \right) dr = \frac{r}{\chi} (1 - e^{-\lambda});$$

for $\Delta \ll 1$ these relations take the form

$$\varkappa \left(\frac{\partial m}{\partial t} \right)_r = \frac{\varkappa r^2}{4\Delta} (\varepsilon + p) - \varkappa r^2 p = 1 - (r e^{-\lambda})_r, \quad (12)$$

$$m = \int_0^r \left[\frac{r^2}{4\Delta} (\varepsilon + p) - r^2 p \right] dr = \frac{\varkappa r}{\varkappa} (1 - e^{-\lambda}).$$

Let us now pass to the independent variables $(m; r)$. The system of equations (10), (11) then takes the form

$$r \frac{\partial \lambda}{\partial m} \frac{\partial \tau}{\partial r} + \frac{\partial \tau}{\partial m} \left[1 - e^{-\lambda} (1 + \kappa r^2 p) + \frac{\kappa r^2}{4\Delta} e^{\lambda} (\varepsilon + p) - r \frac{\partial \lambda}{\partial r} \right] = 0,$$

$$A \frac{\partial \lambda}{\partial m} + \frac{\kappa r e^{\lambda}}{4\Delta} (\varepsilon + p) \frac{\partial \tau}{\partial m} = 0.$$

Since $\lambda = -\ln(1 - \kappa m/r)$, $\partial \lambda / \partial m = \kappa e^{\lambda} / r$; $\partial \lambda / \partial r = -\kappa m e^{\lambda} / r^2$, from these relations we find

$$\partial \tau / \partial m = -4\Delta A / r^2 (p + \varepsilon), \quad \partial \tau / \partial r = A[1 - 4\Delta p / (p + \varepsilon)]. \quad (13)$$

Equations (7), (8), (9) in the variables $(m; r)$ may be written in the form

$$r^2 p x_m = x_r; \quad (14)$$

$$r^2 p y_m = y_r; \quad (15)$$

$$r^2 p \sigma_m = \sigma_r. \quad (16)$$

Write equation (14) in the form

$$e^x r^{4(1-k)/(2-k)} \partial x / \partial m = \partial x / \partial r. \quad (17)$$

Its general solution has the form

$$m = \frac{(2-k)}{(5k-6)} e^x r^{(6-5k)/(2-k)} + F_1(x) = \frac{2-k}{5k-6} pr^3 + F_1(pr^{2k/(2-k)}). \quad (18)$$

Writing the system (14), (15), and (16) in the independent variables $(x; r)$, we find that $y = y(x)$, $\sigma = \sigma(x)$, and, consequently,

$$m = \frac{2-k}{5k-6} pr^3 + F_2(y) = \frac{2-k}{5k-6} pr^3 + F_3(\sigma). \quad (19)$$

At the same time

$$\Delta = r^{-4(k-1)/(2-k)} (1 - \kappa m/r) f_1(pr^{2k/(2-k)}); \quad (20)$$

$$\sigma = f_3(pr^{2k/(2-k)}). \quad (21)$$

On the basis of equation (12) we have

$$\left(\frac{\partial m}{\partial r}\right)_\tau = \frac{r^2(\varepsilon + p)}{4\Delta} - pr^2 = \frac{k}{4(k-1)} \frac{pr^2}{\Delta} - pr^2 = \frac{k}{4(k-1)} e^{x-y+\lambda} - pr^2 \quad (22)$$

or

$$(\partial m / \partial r)_\tau = f(x) / (1 - \varkappa m) r - pr^2, \quad (23)$$

where

$$f(x) = ke^{x-y}/4(k+1); \quad r^2(\varepsilon + p)/4\Delta = f(x)e^\lambda = f(x)/(1 - \varkappa m/r).$$

From (18) we find that

$$\left(\frac{\partial m}{\partial r}\right)_\tau = -pr^2 + \left(\frac{\partial x}{\partial r}\right)_\tau \left[\frac{2-k}{5k-6} e^{x-r^{(6-5k)/(2-k)}} + \frac{dF_1(x)}{dx} \right]. \quad (24)$$

Comparing (23) and (24), we arrive at the result:

$$\left(\frac{\partial x}{\partial r}\right)_\tau = \frac{f(x)}{\left(1 - \frac{\varkappa m}{r}\right) \left(\frac{2-k}{5k-6} e^{x-r^{(6-5k)/(2-k)}} + \frac{dF_1(x)}{dx} \right)}, \quad (25)$$

where

$$\frac{m}{r} = \frac{2-k}{5k-6} e^{x-r^{4(1-k)/(2-k)}} + \frac{F_1(x)}{r}.$$

Formally solving this equation of the form

$$(\partial x / \partial r)_\tau = \theta(x; r), \quad (26)$$

we find

$$\tau = \bar{\tau}[\xi(x; r)], \quad (27)$$

where $\bar{\tau}(\xi)$ is an arbitrary function.

It is now easy to determine the quantity

$$e^{(\lambda-\nu)/2} = A = -\frac{\partial\tau}{\partial m} \frac{r^2(p+\varepsilon)}{4\Delta} = -\frac{\partial\tau}{\partial m} \frac{kpr^2}{4(k-1)\Delta} = \frac{\partial\tau}{\partial m} \frac{f(x)}{1-\kappa m/r}. \quad (28)$$

Since $e^\lambda = 1/(1-\kappa m/r)$, we find

$$e^{-\nu/2} = -\frac{\partial\tau}{\partial m} \frac{f(x)}{\sqrt{1-\kappa m/r}}, \quad (29)$$

and thereby completely solve the problem posed.

Let us now turn to simplifications. For $\kappa = 0$, i.e., in the absence of a gravitational field, we have the solution ⁽²⁾

$$x = x(r-\tau), \quad y = y(r-\tau), \quad \sigma = \sigma(r-\tau), \quad (30)$$

which follows immediately from (7), (8), (9) for $A = 1$.

Neglecting in the second equation (13) the quantity $\Delta \ll 1$, we find that

$$\partial\tau/\partial r = A. \quad (31)$$

In this case equations (14), (15), (16) take the form

$$x = x(m), \quad y = y(m), \quad \sigma = \sigma(m), \quad (32)$$

$$(\partial m/\partial r)_\tau = f(x)/(1-\kappa m/r) = f(m)/(1-\kappa m/r). \quad (33)$$

For $\kappa = 0$

$$(\partial m/\partial r)_\tau = f(m). \quad (34)$$

Integrating, we find that

$$r - \varphi(m) = T(\tau) = \tau,$$

whence $m = \psi(r-\tau)$ and $x = x(r-\tau)$, which gives the limiting results (30).

For $\kappa \neq 0$ it is necessary to integrate equation (33):

$$\frac{dr}{1} = \frac{d\tau}{0} \frac{dm}{f(m)} \left(1 - \frac{\kappa m}{r}\right), \quad (35)$$

whence

$$\frac{dr}{dm} f(m) = 1 - \frac{\kappa m}{r}.$$

Putting $r = 1/\xi$, we shall have

$$\frac{d\xi}{dm} f(m) = \kappa m \xi^3 - \xi^2 \quad (36)$$

or

$$d\xi/d\eta = B(\eta)\xi^3 - \xi^2, \quad (37)$$

where $d\eta = dm/f(m)$; $B(\eta) = \kappa m(\eta)$.

This equation is easily solved by numerical methods (for example, by the Runge-Kutta method); as a result, returning to the old variables, we find

$$D(r; m) = f(\tau), \quad (38)$$

which then permits one formally to express x, y, σ through r and τ .

In the case when $f = f_0 = \text{const}$, and this is an important case, equation (35) is integrated at once:

$$m^2 \left[\frac{r^2}{m^2} - \frac{r}{mf_0} + \frac{\kappa}{f_0} \right] + F(\tau) \left[\frac{r/m - 1/2f_0 + (1/2f_0)\sqrt{1-4\kappa f_0}}{-r/m + 1/2f_0 + (1/2f_0)\sqrt{1-4\kappa f_0}} \right]^{1/\sqrt{1-4\kappa f_0}}. \quad (39)$$

For $\chi = 0$

$$\frac{r}{m} f_0 = \frac{\psi(\tau)}{m} + 1; \quad (40)$$

further, from (39) we have

$$-\frac{m^2}{f_0^2} \left[\frac{\psi^2}{m^2} + \frac{\psi}{m} \right] - F(\tau) \left[\frac{\psi/m + 1}{\psi/m} \right] = 0;$$

whence

$$\frac{m^2}{f_0^2} \frac{\psi^2}{m^2} = F(\tau) = \frac{\psi^2(\tau)}{f_0^2},$$

which verifies the calculations we have carried out. From (20) it follows that in the case

$$f_1 = r_0^{4(k-1)/(2-k)} = \text{const}$$

with

$$\chi m/r = 1, \quad r = r_g(m), \quad \partial r/\partial m = 0 \quad (f(\chi) = f(m) \neq 0).$$

It also follows from this that here $r = r_{\min} = 2Gm_0/c^2 = r_g$, where $m_0 = 4\pi m/c^2$, $\Delta = 0$, i.e., at the gravitational radius $u = c$. As $r \rightarrow \infty$, also $\Delta = 0$, $u = c$. The maximum value $\Delta = \Delta_{\max}$, i.e., $u = u_{\min}$, is attained at

$$r = \frac{3k-2}{4(k-1)} \chi^m = \frac{3k-2}{4(k-1)} \frac{2Gm_0}{c^2} = \frac{3k-2}{4(k-1)} r_g.$$

Thus, in the ultrarelativistic limiting case there is a radius $r = r^*$ at which the velocity is minimal; this, however, still does not mean that the contraction of the medium stops.

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CITED LITERATURE

1. K. P. Stanyukovich, DAN, 182, No. 2 (1968). 2. K. P. Stanyukovich, ZhETF, 36, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.

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