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Abstract

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MATHEMATICS

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ON ASYMPTOTICALLY EFFICIENT ESTIMATION OF REGRESSION COEFFICIENTS

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Let $\xi(t) = \theta(t) + \eta(t)$ be a Gaussian process with discrete time, representing the sum of a real function $\theta(t) = \alpha_1\theta_1(t) + \dots + \alpha_p\theta_p(t)$ from the linear span of known functions $\theta_1(t), \dots, \theta_p(t)$ and a real stationary process $\eta(t)$ with zero mean. It is required to estimate the unknown coefficients $\alpha_1, \dots, \alpha_p$ from observation of the process $\xi(t)$ on the interval $0 \leq t \leq n$. As is known, for the parameters $\alpha_1, \dots, \alpha_p$ there exist best unbiased estimates, coinciding with maximum-likelihood estimates (see, for example, ⁽¹⁾). However, constructing the best estimates requires knowledge of the correlation function $B(t)$ (or the spectral measure $F(d\lambda)$) of the process $\eta(t)$; moreover, the corresponding computations are connected with inversion of the matrix $\{B(t-s), t, s = 0, 1, \dots, n\}$, and this substantially complicates the practical use of the best estimates for large n . In the present work a new class of unbiased estimates for the coefficients $\alpha_1, \dots, \alpha_p$, proposed by one of the authors in ⁽²⁾, is considered. Namely, as estimates $\hat{\alpha}_1, \dots, \hat{\alpha}_p$ one proposes maximum-likelihood estimates computed under the assumption that the Gaussian stationary process $\eta(t)$ has as its spectral density an a priori chosen function $g(\lambda)$. In what follows $g(\lambda)$ will be called a pseudospectral density, and the corresponding estimates $\hat{\alpha}_1, \dots, \hat{\alpha}_p$ pseudobest. Below the asymptotic properties of pseudobest estimates with respect to the true probability distribution (corresponding to the true spectral density $f(\lambda)$) are studied.

Let us indicate explicit expressions for the pseudobest estimates. Let

$$\psi_j(\lambda) = \sum_{0 \leq s \leq n} c_j(s) e^{i\lambda s}, \quad j = 1, \dots, p,$$

be solutions of the equations

$$\int e^{-i\lambda t} \psi_j(\lambda) g(\lambda) d\lambda = \theta_j(t), \quad 0 \leq t \leq n. \quad (1)$$

Denote by $\{\psi_j^*, j = 1, \dots, p\}$ the system conjugate to $\{\psi_j, j = 1, \dots, p\}$ in the Hilbert space $L^2(g)$ of functions on the interval $[-\pi, \pi]$ with scalar product

$\langle \varphi, \psi \rangle_g = \int \varphi(\lambda) \overline{\psi(\lambda)} g(\lambda) d\lambda$. The observed random process is harmonizable ($\xi(t) = \int e^{i\lambda t} \Phi(d\lambda)$) with stochastic measure $\Phi(d\lambda) = \Psi(d\lambda) + [\alpha_1 \psi_1(\lambda) + \dots + \alpha_p \psi_p(\lambda)] \times g(\lambda) d\lambda$, where $\Psi(d\lambda)$ is the stochastic spectral measure of the process $\eta(t)$. The pseudobest estimates can be represented in the following spectral form:

$$\hat{\alpha}_j = \int \psi_j^*(\lambda) \Phi(d\lambda), \quad j = 1, \dots, p. \quad (2)$$

We note that for $g(\lambda) \equiv 1$ we obtain the classical least-squares estimates, and for $g(\lambda) \equiv f(\lambda)$, the truly best unbiased estimates.

From a practical point of view, the most interesting case is when pseudospectral density is chosen in the form

$$g(\lambda) = \frac{1}{2\pi} |Q(e^{i\lambda})|^{-2}, \quad Q(z) = \sum_{0 \leq k \leq m} g_k z^k, \quad (3)$$

where the polynomial (with real coefficients) $Q(z)$ has no zeros in the disk $|z| \leq 1$. Let Θ_n be the $(n+1) \times p$ -matrix whose columns are the values of the functions $\theta_1(t), \dots, \theta_p(t)$, $0 \leq t \leq n$. Introduce the shift operator Δ ($\Delta\theta(t) = \theta(t+1)$) and put

$$Q(\Delta) = \sum_{0 \leq k \leq m} q_k \Delta^k, \quad Q(\Delta^{-1}) = \sum_{0 \leq k \leq m} q_k \Delta^{-k}.$$

Then

$$\hat{\alpha}^{(n)} = \{\Theta_n^* [Q(\Delta) Q(\Delta^{-1}) \Theta_n]\}^{-1} [Q(\Delta) Q(\Delta^{-1}) \Theta_n] \xi^{(n)}, \quad (4)$$

where $\hat{\alpha}^{(n)}$ denotes the vector $(\hat{\alpha}_1, \dots, \hat{\alpha}_p)$, $\xi^{(n)} = (\xi(0), \xi(1), \dots, \xi(n))$, and the operator $Q(\Delta) Q(\Delta^{-1})$ is applied to the functions $\theta_j(t)$, previously continued from the interval $0 \leq t \leq n$ to the whole axis by formulas (1). We note that for $n \geq m$ the indicated continuation is easily carried out by means of linear recurrence relations:

$$\begin{aligned} \theta(-j) &= -\frac{1}{q_0} \sum_{1 \leq k \leq m} q_k \theta(k-j), \\ \theta(n+j) &= -\frac{1}{q_0} \sum_{1 \leq k \leq m} q_k \theta(n+j-k), \quad j > 0. \end{aligned}$$

From expression (4) it is easy to see that the number of operations necessary for computing $\hat{\alpha}^{(n)}$ (for fixed n) is a linear function of n . If, however, the

observations are made sequentially, then it is expedient at each computational step to keep in memory the $p \times p$ -matrix $\Theta_n^*[Q(\Delta)Q(\Delta^{-1})\Theta_n]$ and the p -vector $[Q(\Delta)Q(\Delta^{-1})\Theta_n]\xi^{(n)}$. In this case, increasing the observation interval by 1 leads to additional computations of bounded complexity (depending only on p), and as a result the number of operations of such a computational process is also a linear function of n .

We shall assume that the true spectral density $f(\lambda)$ and the chosen pseudospectral density $g(\lambda)$ are continuous and positive, and that the functions $\theta_1(t), \dots, \theta_p(t)$ satisfy the conditions:

$$\sum_{0 \leq t \leq n} |\theta_j(t)|^2 \rightarrow \infty, \quad n \rightarrow \infty, \quad j = 1, \dots, p; \quad (5)$$

$$\max_{0 \leq t \leq n} |\theta_j(t)| = o \left(\sum_{0 \leq t \leq n} |\theta_j(t)|^2 \right)^{1/2}, \quad (6)$$

for all s there exists the limit

$$\lim_{n \rightarrow \infty} \sum_{0 \leq t \leq n} \theta_k(t+s)\theta_j(t) / \left(\sum_{0 \leq t \leq n} |\theta_k(t)|^2 \sum_{0 \leq t \leq n} |\theta_j(t)|^2 \right)^{1/2} = R_{kj}(s),$$

$$\det\{R_{kj}(0)\} > 0. \quad (7)$$

As is known, the matrix function $R(s)$ admits the spectral representation $R(s) = \int e^{i\lambda s} H^0(d\lambda)$ with a positive definite matrix measure $H^0(d\lambda) = \{H_{kj}^0(d\lambda)\}$ (usually called the regression measure, see (1)). Put $H(d\lambda) = \frac{1}{g(\lambda)} H^0(d\lambda)$, $H = \int H(d\lambda)$. In addition, introduce the normalizing diagonal matrix D_n with elements

$$\left(\sum_{0 \leq t \leq n} |\theta_j(t)|^2 \right)^{1/2}$$

on the main diagonal.

Theorem 1. For the correlation matrix $\{\sigma_{kj}^{(n)}\}$ of the pseudobest

of the estimates $\hat{\alpha}_1^{(n)}, \dots, \hat{\alpha}_p^{(n)}$ the relation holds

$$\lim_{n \rightarrow \infty} D_n \{\sigma_{kj}^{(n)}\} D_n = 2\pi H^{-1} \int \frac{f(\lambda)}{g(\lambda)} H(d\lambda) H^{-1}. \quad (8)$$

Theorem 1 makes it possible to formulate conditions for the asymptotic efficiency of pseudo-best estimates, similarly to how this is done in (1) for the case of least-squares estimates ($g(\lambda) \equiv 1$). Namely, equating the asymptotic expressions (8) for the correlation matrices of the best ($g(\lambda) \equiv f(\lambda)$) and pseudo-best estimates, we arrive at the following theorem:

Theorem 2. *For the asymptotic efficiency of pseudo-best estimates corresponding to the pseudospectral density $g(\lambda)$, it is necessary and sufficient that the matrix function $\frac{f(\lambda)}{g(\lambda)}E$ (where E is the identity $p \times p$ matrix) be constant almost everywhere with respect to the matrix measure*

$$\tilde{H}(d\lambda) = H^{-1/2}H(d\lambda)H^{-1/2},$$

i.e.

$$\left[\frac{f(\lambda)}{g(\lambda)}E - C \right] \tilde{H}(d\lambda) = 0$$

for some constant matrix $C = \{c_{kj}\}$.

It is easy to see that, with a proper choice of $g(\lambda)$, the estimates $\hat{\alpha}_1^{(n)}, \dots, \hat{\alpha}_p^{(n)}$ are asymptotically efficient, whereas ordinary least-squares estimates ($g(\lambda) \equiv 1$), generally speaking, do not possess this property. We emphasize that in the practically most important case, when the measure $H^0(d\lambda)$ is concentrated at a finite number of points (for example, when $\theta_j(t) = P_j(t) \int e^{i\lambda t} m_j(d\lambda)$, where $P_j(t)$ are polynomials and $m_j(t)$ are finite measures with a finite number of jumps), in order to obtain asymptotically efficient estimates, one may take as the pseudospectral density a function $g(\lambda)$ of type (3).

We give below the proof of Theorem 1.

Lemma 1. *If the matrix measure*

$$H^{(n)}(d\lambda) = \left\{ \frac{\psi_k(\lambda)\overline{\psi_j(\lambda)}}{\|\psi_k\|_g\|\psi_j\|_g} g(\lambda) d\lambda \right\} \quad (\|\psi\|_g^2 = \langle \psi, \psi \rangle_g) \quad (9)$$

converges weakly as $n \rightarrow \infty$ to some measure $\tilde{H}(d\lambda)$, then

$$\lim_{n \rightarrow \infty} \tilde{D}_n \{\sigma_{kj}^{(n)}\} \tilde{D}_n = \tilde{H}^{-1} \int \frac{f(\lambda)}{g(\lambda)} \tilde{H}(d\lambda) \tilde{H}^{-1},$$

where $\tilde{H} = \int \tilde{H}(d\lambda)$, and $\tilde{D}_n$ is the diagonal matrix with elements $\|\psi_j\|_g$ on the main diagonal.

The proof follows easily from the following relation for the correlation matrix $\{\sigma_{kj}^{(n)}\}$ of the pseudo-best estimates:

$$\{\sigma_{kj}^{(n)}\} = \{\langle \psi_k, \psi_j \rangle_f\} = \{\langle \psi_k, \psi_j \rangle_g\}^{-1} \{\langle \psi_k, \psi_j \rangle_f\} \{\langle \psi_k, \psi_j \rangle_g\}^{-1}.$$

Lemma 2. For $g(\lambda)$ of the form (3), the matrix measure (9) converges weakly to

$$\{\tilde{H}_{kj}(d\lambda)\} = \{H_{kk}^{-1/2} H_{kj}(d\lambda) H_{jj}^{-1/2}\}.$$

For the proof one must use the fact that the solutions of equations (1) can be represented in the form (see (3))

$$\psi_j^{(n)}(\lambda) = \tilde{\theta}_j^{(n)}(\lambda) |Q(e^{i\lambda})|^2 + R_j^{(n)}(\lambda),$$

where

$$\tilde{\theta}_j^{(n)}(\lambda) = \sum_{0 \leq t \leq n} e^{i\lambda t} \theta_j(t), \quad \max |R_j^{(n)}(\lambda)| = o \left(\sum_{0 \leq t \leq n} |\theta_j(t)|^2 \right)^{1/2}.$$

From Lemmas 1 and 2 follows the validity of Theorem 1 for functions $g(\lambda)$ having the special form (3). To obtain the assertion of the theorem in the general case, it suffices to show that the matrix measure (9) for an arbitrary continuous positive $g(\lambda)$ converges weakly. The latter result is derived by uniformly approximating $g(\lambda)$ from below by functions of the form (3).

Let $g(\lambda)$ and $g'(\lambda)$ be continuous, positive, $0 < m \leq g'(\lambda) \leq g(\lambda) \leq M$, and let

$$\psi_j(\lambda) = \sum c_j(s) e^{i\lambda s}, \quad \psi'_j(\lambda) = \sum c'_j(s) e^{i\lambda s}$$

be the corresponding solutions of equations (1), i.e., for $0 \leq t \leq n$,

$$\int e^{-i\lambda t} \psi'_j(\lambda) g'(\lambda) d\lambda = \int e^{-i\lambda t} \psi_j(\lambda) g(\lambda) d\lambda \quad (= \theta_j(t)) \quad (10)$$

or

$$\sum c'_j(s) b'(t-s) = \sum c_j(s) b(t-s) = \theta_j(t), \quad (11)$$

where $2\pi b(t)$, $2\pi b'(t)$ are the Fourier coefficients of the functions $g(\lambda)$, $g'(\lambda)$.

To estimate the discrepancy of the matrix measures (9) corresponding to $g(\lambda)$ and $g'(\lambda)$, consider

$$\int e^{i\lambda t} [\psi_j(\lambda) \overline{\psi_k(\lambda)} g(\lambda) - \psi'_j(\lambda) \overline{\psi'_k(\lambda)} g'(\lambda)] d\lambda. \quad (12)$$

Adding and subtracting from (12) the expressions $\psi_j(\lambda)\overline{\psi'_k(\lambda)}g(\lambda)$ and $\psi_j(\lambda)\overline{\psi'_k(\lambda)}g'(\lambda)$, we split the integral (12) into three parts, each of which is estimated separately using Cauchy's inequality and the relations (10); the latter are applied after replacing $\psi'_k(\lambda)$ (or $\psi_j(\lambda)$) by their representation as linear combinations of $e^{i\lambda s}$. As a result, the modulus of the integral (12) is bounded above by

$$M\left(\sum^* |c_j(s)|^2\right)^{1/2} (\|\psi_k\| + \|\psi'_k\|) + M\left(\sum^* |c'_k(s)|^2\right)^{1/2} (\|\psi_j\| + \|\psi'_j\|) + \\ + \left(1 + \frac{1}{\sqrt{2\pi}}\right) \max |g(\lambda) - g'(\lambda)| (\|\psi_k\| + \|\psi'_k\|) (\|\psi_j\| + \|\psi'_j\|), \quad (13)$$

where the summation $\sum^*$ is over all s such that $0 \leq s \leq t-1$, $n-t+1 \leq s \leq n$, and

$$\|\psi\|^2 = \int |\psi(\lambda)|^2 d\lambda.$$

Passing in the right-hand equality (11) to the matrix inverse to $\{b(t-s)\}$, and using Cauchy's inequality, it is easy to show that

$$\left(\sum |c_k(s) - c'_k(s)|^2\right)^{1/2} \leq \frac{1}{m^2} [g(\lambda) - g'(\lambda)] \left(\sum |c_k(s)|^2\right)^{1/2}. \quad (14)$$

It is somewhat more difficult to prove that, for each fixed t ,

$$c_j(t), \quad c_j(n-t) = o\left(\|\tilde{\theta}_j^{(n)}\|\right). \quad (15)$$

Together with (14) and (15), estimate (13) makes it possible to establish the existence of the corresponding limit for the matrix measure (9).

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