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Abstract

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MATHEMATICS

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ESTIMATES OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS WITH THE BESSEL OPERATOR

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In the modern theory of boundary-value problems, an important place is occupied by the question of limiting sharp estimates of the norms of solutions of boundary-value problems in terms of the norms of the right-hand sides and boundary conditions. The present note is devoted mainly to estimates of solutions of boundary-value problems for certain singular differential equations. These estimates can be obtained rather simply, once explicit formulas are known for the solutions of equations with constant coefficients.

In the metrics $L_{2,k}$, estimates of solutions were obtained earlier by I. A. Kipriyanov in the paper ⁽¹⁾.

1. Let E_{n+2}^{++} be the Euclidean space of points (x, y, t) , $x = (x_1, \dots, x_n)$, $y \geq 0$, $t \geq 0$. In this domain we shall consider a solution $u(x, y, t)$ of the B -elliptic ⁽¹⁾ equation of order $2m$ with constant (complex) coefficients

$$\mathcal{L}(D, B_y)u = \sum_{i+2r=2m} a_{i,r} D^i B_y^r u = f(x, y, t) \quad \text{for } t > 0, \quad (1)$$

satisfying m boundary conditions

$$H_j(D, B_y)u = \sum b_{i,r}^j D^i B_y^r u = g_j(x, y) \quad \text{for } t = 0. \quad (2)$$

These conditions are expressed by differential operators H_j with constant coefficients. The order of the operator H_j is equal to m_j . The numbers m_j are nonnegative and may exceed the number $2m$ —the order of the operator $\mathcal{L}$. In addition, $\mathcal{L}$ and all H_j are homogeneous. Here

$$B_y = - \left(\frac{\partial^2}{\partial y^2} + \frac{k}{y} \frac{\partial}{\partial y} \right), \quad k > 0,$$

$$D = (D_x, D_t), \quad \text{where} \quad D_t = \frac{1}{i} \frac{\partial}{\partial t}.$$

The functions f and g_j for $t \geq 0$ and $y \geq 0$ are infinitely differentiable and have compact support. In what follows we shall assume that the boundary operators are connected with the operator $\mathcal{L}$ by the Ya. B. Lopatinskii condition (the complementing condition ⁽²⁾).

For sufficiently large N , let C_B^N denote the class of functions continuously differentiable N times with respect to x and t , and continuously admitting $N/2$ applications of the Bessel operator with respect to the variable y in the case of even N , and $\frac{\partial}{\partial y} B_y^{(N-1)/2}$ in the case of odd N . Extend the function f to the entire half-space $y \geq 0$ so that $f_N \in C_B^N$. The solution of equation (1) with right-hand side f_N has ⁽³⁾ the form

$$v = v_N(x, y, t) = \mathcal{E} * f_N = \int T_y^z \mathcal{E}(x - s, y, t - \tau) f_N(s, z, \tau) z^k ds dz d\tau, \quad (3)$$

where $\mathcal{E}$ is the fundamental solution of the B -elliptic equation, determined by the formula ⁽³⁾

$$\mathcal{E}(P) = |P|^{2m-n-k-2} \Omega\left(\frac{P}{|P|}\right) + q(P) \ln |P|, \quad P = (x, y, t),$$

and T_y^z is the generalized shift operator

$$T_y^z f(y) = \frac{\Gamma((k+1)/2)}{\Gamma(1/2)\Gamma(k/2)} \int_0^\pi f(\sqrt{y^2 + z^2 - 2yz \cos \alpha}) \sin^{k-1} \alpha d\alpha. \quad (4)$$

Integration in formula (3) is carried out over the whole half-space $z > 0$.

Let $u = v_N + w$. To determine the function w we obtain the following problem: $\mathcal{L}w = 0$ for $t > 0$ and $H_{jw} = \varphi_j(x, y)$ for $t = 0$, where the function

$$\varphi_j(x, y) = g_j(x, y) - H_j v_N|_{t=0} \quad (5)$$

is no longer finite. Using the results of paper ⁽⁴⁾, for the solution of the latter problem we obtain the representation

$$D^i B_y^r w = \sum_{j=1}^m \int_{E_{n+1}^+} D^i B_y^r T_y^z K_j(x - s, y, t) \varphi_j(s, z) z^k ds dz, \quad (6)$$

where $i + 2r = l_0 = \max(2m, m_j)$, K_j is the Poisson kernel of the B -elliptic problem constructed in ⁽⁴⁾, so that for the finite function u we obtain the representation

$$D^i B_y^r u = D^i B_y^r v_N + D^i B_y^r w. \quad (7)$$

The uniqueness of the function u defined by (7), (6), can be proved according to the scheme of the proof of Theorem 4.1 of paper ⁽²⁾.

2. Let l be an arbitrary integer $\geq l_0$. For any function v of the class C_B^l put

$$[v]_l = \sup |D^i B_y^r v|, \quad |v|_l = \sum_{j=0}^l [v]_j,$$

where $i + 2r = l$. The subclass of all functions from C_B^l for which the expressions $D^i B^r v$ uniformly satisfy the Hölder condition with exponent α ($0 < \alpha < 1$) will be denoted by $C_B^{l+\alpha}$. For these functions we define the seminorm by the formula

$$[v]_{l+\alpha} = \sup \frac{|D^i B^r v(P) - D^i B^r v(Q)|}{|P - Q|^\alpha},$$

where the supremum is taken over all $i + 2r = l$ and $P \neq Q$.

Theorem 1. Let $u(x, y, t)$ be a solution of problem (1)–(2), belonging to the class $C_B^{l_0+\alpha}$, and let it have compact support. If $f \in C_B^{l-2m+\alpha}$ in the quarter-space $y > 0$, $t > 0$, and $g_j \in C_B^{l-m_j+\alpha}$ on the half-plane $t = 0$.

Then $u \in C_B^{l+\alpha}$,

$$[u]_{l+\alpha} \leq C \left([f]_{l-2m+\alpha} + \sum [g_j]_{l-m_j+\alpha} \right), \quad (8)$$

where C depends only on l, α and on the B -ellipticity constant of our problem.

Let $\Sigma^+ = \Sigma_R^+$ denote the quarter-ball:

$$|x|^2 + y^2 + t^2 < R^2, \quad y \geq 0, \quad t \geq 0$$

in (x, y, t) -space. Denote by σ_R^+ the flat part of the boundary: $t = 0$, $y \geq 0$; d_P is the distance from the point P in Σ^+ to the spherical part of the boundary Σ^+ , and $d_{P,Q} = \min(d_P, d_Q)$. By $\widehat{C}_B^l(\Sigma^+)$ denote the class of functions u for which the norm is finite

$$|\widehat{u}|_{l+\alpha} = \sum_{j=0}^l [\widehat{u}]_j + [\widehat{u}]_{l+\alpha},$$

where

$$\widehat{[u]}_j = \sup d_P^j |D^i B_y^r u|,$$

$$\widehat{[u]}_{l+\alpha} = \sup_{P,Q} d_{P,Q}^{l+\alpha} \frac{|D^i B^r u(P) - D^i B^r u(Q)|}{|P - Q|^\alpha}.$$

Theorem 2. Let u be a bounded solution of problem (1)–(2), belonging to the class $C_B^{l_0+\alpha}$. Suppose, moreover, that $f \in \widehat{C}_B^{l-2m+\alpha}(\Sigma^+)$ and $g_j \in \widehat{C}_B^{l-m_j+\alpha}(\sigma^+)$ for fixed $l \geq l_0$.

Then the function $u \in C_B^{l+\alpha}$ and the inequality holds

$$[\widehat{u}]_{l+\alpha} \leq C \left(d_{P,Q}^{2m} [\widehat{f}]_{l-2m+\alpha} + \sum d_{P,Q}^{m_j} [\widehat{g}_j]_{l-m_j+\alpha} + [\widehat{u}]_0 \right), \quad (9)$$

where C does not depend on u, f, g_j , and R .

If growth of the solution at infinity with some rate is allowed, then Theorem 2 implies a generalization of Theorem 1.

Theorem 3. Suppose the quantity

$$M_0 = \lim_{R \rightarrow \infty} R^{-(l+\alpha)} \max_{\Sigma^+} |u|$$

is finite.

Then $u \in C_B^{l+\alpha}$ and the inequality holds

$$[u]_{l+\alpha} \leq C \left([f]_{l-2m+\alpha} + \sum [g_j]_{l-m_j+\alpha} + M_0 \right). \quad (10)$$

3. The theorems given above make it possible, by means of the usual procedure of “freezing” the coefficients, to obtain analogues of Theorems 1 and 2 for equations with variable coefficients. In this case our equations have the form

$$\mathcal{L}(P, D, B_y)u(P) \equiv \sum_{i+2r \leq 2m} a_{i,r}(P) D^i B_y^r u(P) = F(P), \quad t > 0, \quad (11)$$

$$H_j(P', D, B_y)u(P) = \sum_{i+2r \leq m_j} b_{i,r}^j(P') D^i B_y^r u(P) = G_j(P'), \quad t = 0, \quad (12)$$

where $P = (x, y, t)$, $P' = (x, y, 0)$.

We also assume that the operators $\mathcal{L}, H_j$ and their coefficients satisfy certain conditions of type i)–iii) of work ⁽²⁾ (p. 70). We shall not formulate them, but note that they are essential.

Theorem 4. Let the function u belong to the class $C_B^{l+\alpha}$ and be a solution of the boundary-value problem (11)–(12) in the domain E_{n+2}^{++} .

Then

$$|u|_{l+\alpha} \leq C \left([F]_{l-2m+\alpha} + \sum [G_j]_{l-m_j+\alpha} + |u|_0 \right), \quad (13)$$

where C does not depend on u, F , and G_j .

Theorem 5. Let u be a solution of problem (11)–(12), belonging to the class $C_B^{l_0+\alpha}$ in Σ^+ for $R < 1$.

Then $u \in C_B^{l+\alpha}(\Sigma^+)$ and

$$|\hat{u}|_{l+\alpha} \leq C \left(d_{P,Q}^{2m} [\widehat{F}]_{l-2m+\alpha} + \sum d_{P,Q}^{m_j} [\widehat{G}_j]_{l-m_j+\alpha} + |\hat{u}|_0 \right), \quad (14)$$

where C does not depend on R .

4. In conclusion we give $L_{p,k}$ -estimates for solutions of problems (1)–(2) and (11)–(12). Put, for $p > 1$ ($t = x_{n+2}$),

$$[u]_{j,L_{p,k}} = \left(\sum_{i=1}^n \int_{E_{n+2}^{++}} |D_{x_i}^j u|^p y^k dx dy + \int_{E_{n+2}^{++}} |D_t^j u|^p y^k dx dy + [B_y^{[j/2]} u]_{y,j-2[j/2],L_{p,k}}^p \right)^{1/p},$$

$$\|u\|_{l,L_{p,k}} = \left(\sum_{j=0}^l [u]_{j,L_{p,k}}^p \right)^{1/p}. \quad (15)$$

where

$$[B_y^{[j/2]} u]_{y,j-2[j/2],L_{p,k}} = \left(\int_{E_{n+2}^{++}} |B_y^{j/2} u|^p y^k dx dy \right)^{1/p}$$

in the case of even j . If j is odd ($j = 2r + 1$), then

$$[B_y^{[j/2]} u]_{y,1,L_{p,k}} = \left(\int_{E_{n+2}^{++}} y^k dx dy \int_0^\infty \frac{|T_y^{2h} B^r u - 2T_y^h B^r u + B^r u|^p}{h^{1+p}} dh \right)^{1/p}.$$

For functions $\varphi(x, y)$ that are boundary values of functions $u(x, y, t)$, and for positive integers l , define the norms

$$\|\varphi\|_{l-1/p, L_{p,k}} = \inf \|u\|_{l, L_{p,k}},$$

where the infimum is taken over all such functions u that $u(x, y, 0) = \varphi(x, y)$.

Theorem 6. Let u be a solution of problem (1)–(2) and let it be finite. Suppose, moreover, that $\|u\|_{l, L_{p,k}} < \infty$, $l \geq l_0 + 1$.

Then

$$\|u\|_{l, L_p, k} \leq C \left(\|f\|_{l-2m, L_p, k} + \sum \|g_j\|_{l-m_j-1/p, L_p, k} \right), \quad (16)$$

where the constant C depends only on l, p , and the constant of B -ellipticity of our problem.

Theorem 7. Let u be a solution of problem (11)–(12) and tend to zero as $|P| > \rho$, where ρ is sufficiently small. Suppose that the norm $\|u\|_{l_0+1, L_p, k}$ is finite.

Then the norm $\|u\|_{l, L_p, k}$ for $l \geq l_0 + 1$ is also finite and

$$\|u\|_{l, L_p, k} \leq C \left(\|F\|_{l-2m, L_p, k} + \sum \|G_j\|_{l-m_j-1/p, L_p, k} + \|u\|_{0, L_p, k} \right), \quad (17)$$

where C does not depend on u, F, G_j .

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References Cited

1. I. A. Kipriyanov, DAN, **158**, No. 2 (1964).
2. S. Agmon, A. Douglis, L. Nirenberg, *Estimates of Solutions of Elliptic Equations Near the Boundary*, IL, 1962.
3. V. I. Kononenko, DAN, **172**, No. 2 (1967).
4. I. A. Kipriyanov, M. I. Klyuchantsev, DAN, **183**, No. 5 (1968).

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