

ON THE PROBLEM OF THE GRAVITATIONAL STABILITY OF A DUST CLOUD

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Abstract

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MATHEMATICAL PHYSICS

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ON THE PROBLEM OF THE GRAVITATIONAL STABILITY OF A DUST CLOUD

(Presented by Academician Ya. B. Zel'dovich, 1 X 1968)

0. Statement of the problem. We consider the equation

$$\partial f / \partial t + v^k \partial f / \partial x^k + F^k(x) \partial f / \partial v^k = 0; \quad f = f(x, v, t); \quad x = (x^1, x^2, x^3);$$

$$v = (v^1, v^2, v^3); \quad F^k(x) = -\frac{\partial \Phi}{\partial x^k}; \quad \Phi(x) = -\gamma \iint \frac{f(y, v)}{|x - y|} dy dv. \quad (1)$$

The stability of a stationary solution of equation (1) is studied. We shall restrict ourselves to considering the simplest stationary solutions of the form

$$f^0(x, v) = h^0(\varepsilon^0(|x|, |v|)); \quad \varepsilon^0(|x|, |v|) = |v|^2/2 + \Phi^0(|x|). \quad (2)$$

§ 1. Linear consideration.

1. Linearization.

Putting $f = f^0 + \varepsilon g$, we obtain

$$\partial g / \partial t + \hat{A}g - \hat{B}g = 0, \quad (3a)$$

where

$$\hat{A}g = v^k \frac{\partial g}{\partial x^k} + F^{0k}(x) \frac{\partial g}{\partial v^k}; \quad \hat{B}g = \gamma \frac{\partial f^0}{\partial v^k} \frac{\partial}{\partial x^k} \left(\iint \frac{g(y, v)}{|x - y|} dy dv \right). \quad (3b)$$

Following V. A. Antonov ⁽¹⁾, we put $g = g_+ + g_-$; $g_+ = \frac{1}{2}(g(x, v, t) + g(x - v, t))$. Since $\hat{B}g_- = 0$, the consequence of (3) is the following equation for $\psi = g_-$

$$\partial^2 \psi / \partial t^2 = \hat{K}\psi, \quad \text{where} \quad \hat{K} = -\hat{A}^2 + \hat{B}\hat{A}. \quad (4)$$

2. The condition $h^{0'}(\varepsilon) < 0$ and the quadratic form K . Let $h'(\varepsilon^0) < 0$ in the domain Ω , where $h(\varepsilon^0) = f^0(x, v) \neq 0$. The operator $\hat{K}$ is self-adjoint (see ⁽¹⁾) in the scalar product

$$(\psi_1, \psi_2) = \int_{\Omega} \frac{\psi_1(x, v)\psi_2(x, v)}{|h'(\varepsilon^0(x, v))|} dx dv.$$

(The functions ψ are considered only on Ω .) For stability of equation (4) when $h'(\varepsilon) < 0$, it is necessary and sufficient that

$$K(\psi, \psi) = (\hat{K}\psi, \psi) \geq 0. \quad (5)$$

More precisely: condition (5) guarantees the absence of solutions of (4) and (3) growing faster than t . In equation (3) there are obvious growing solutions corresponding to uniform translational motion of the cloud (see ⁽¹⁾). Conditions (5) are insufficient to guarantee the absence of other growing solutions of equation (3). Below, when speaking of stability, we have in mind the nonnegativity of $K(\psi, \psi)$. The difficult question of linearly growing solutions of (3) is left open here.

§ 3. On the simplest sufficient condition of nonnegativity

$$K(\psi, \psi) = \Gamma(\beta, \beta) =$$

$$= - \int \frac{\beta^2(x, v)}{h'(\varepsilon^0(x, v))} dx dv - \gamma \int \frac{\beta(x_1, v_1)\beta(x_2, v_2)}{|x_1 - x_2|} dx_1 dv_1 dx_2 dv_2, \quad (6)$$

where

$$\beta = A\psi = v^k \partial \psi / \partial x^k + F^{0k}(x) \partial \psi / \partial v^k; \quad \psi(x, v) = -\psi(x, -v). \quad (7)$$

Sufficient conditions for the nonnegativity of $K(\psi, \psi)$ are obtained from the requirement $\Gamma(\beta, \beta) \geq 0$ on a class of functions β broader than (7). The crudest sufficient condition is obtained if one requires $\Gamma(\beta, \beta) \geq 0$ for all β . Fix $\int \beta(x, v) dv = \alpha(x)$ and solve the variational problem for the minimum of $\Gamma(\beta, \beta)$ under this additional condition. We obtain

$$\begin{aligned} \min \Gamma(\beta, \beta) &= G(\alpha, \alpha) = \int \frac{\alpha^2(x)}{a^0(x)} dx - \gamma \int \frac{\alpha(x)\alpha(y)}{|x - y|} dx dy = \\ &= \int \left[\frac{\alpha(x)}{a^0(x)} + 4\pi\gamma\Delta^{-1}\alpha(x) \right] \alpha(x) dx^*, \end{aligned} \quad (8a)$$

where

$$a^0(x) = - \int h'(\varepsilon^0(x, v)) dv = \frac{\rho^{0'}(r)}{\Phi^{0'}(r)}; \quad \rho^0(r) = \int f_-^0(x, v) dv; \quad |x| = r. \quad (8)$$

The condition $G(\alpha, \alpha) \geq 0$ for all α is the simplest sufficient condition for the nonnegativity of $K(\psi, \psi)$. It can be rewritten as the condition of nonnegativity of the operator $1/a^0 + 4\pi\gamma\Delta^{-1} \geq 0$, or in the equivalent form

$$\hat{P}_0 = -\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - 4\pi\gamma a^0(r) \geq 0^{**} \quad (9)$$

(see (2)). Let us now note that the equation $\hat{P}_1 \alpha = 0$, where $\hat{P}_1 = \hat{P}_0 + 2/r^2$, has the solution

$$\alpha^0(r) = \frac{1}{r^2} \int_0^r \rho^0(r) r^2 dr.$$

Since

$$\int_0^\infty (\alpha^0(r))^2 r^2 dr < \infty,$$

$\lambda = 0$ is an eigenvalue of $\hat{P}_1$. Since $\hat{P}_0 < \hat{P}_1$, condition (9) is never satisfied. It is not difficult to indicate other sufficient conditions; however, their effectiveness needs to be checked.

Remark. Since $a^0(r) \neq 0$ for $r > 0$, by virtue of the known oscillation theorems (see, for example, (3)), $\hat{P}_1 \geq 0$.

§ 4. Radial perturbations

The function $\bar{\Phi}(x, v)$, invariant under a simultaneous rotation in the spaces x and v , is represented in the form

$$\bar{\psi}(x, v) = \bar{\psi}(|x|, |v|, (x, v)); \quad (x, v) = \sum_1^3 x^k v^k. \quad (10)$$

For brevity we shall agree to call such functions radial. An arbitrary function

$$\psi(x, v) = \bar{\psi}(x, v) + \tilde{\psi}(x, v), \quad (11)$$

where $\bar{\psi}(x, v)$ is radial, and $\int \tilde{\psi}(x, v) \mu(x, v) dx dv = 0$ for any radial μ . If one restricts oneself to considering radial perturbations, then one may require $K(\psi, \psi) > 0^{***}$.

* Δ^{-1} is the operator inverse to the Laplace operator in all space.

** If P and Q are two self-adjoint operators and $Q > 0$, then from $P > Q$ it follows that $P^{-1} < Q^{-1}$. The proof follows easily from writing $P - Q = P^{1/2}(1 - C)P^{1/2}$, $P^{-1} - Q^{-1} = P^{-1/2}(1 - C^{-1})P^{-1/2}$, where $C = P^{-1/2}QP^{-1/2}$.

*** Considering $\psi = -\psi(|x|, |v|)$ does not make sense, since such functions are not invariant with respect to (3).

p. 5. Lemma. Let $\alpha(x) = \int \beta(x, v) dv$, and suppose that in the expansion of $\alpha(x)$ in spherical harmonics there is no zeroth term:

$$\alpha(x) = \sum_{l=1}^{\infty} \alpha_{lm}(r) Y_{lm}(\theta, \varphi). \quad (12)$$

$$|m| < l$$

Then $\Gamma(\beta, \beta) > 0$.

Proof. The minimum of $\Gamma(\beta, \beta)$ under the condition $\int \beta(x, v) dv = \alpha(x)$ is given by formula (8). Substituting (12) into (8), we obtain

$$\min \Gamma(\beta, \beta) = G(\alpha, \alpha) = \sum_{l=1}^{\infty} G_l(\alpha_{lm}, \alpha_{lm}), \quad (13a)$$

$$|m| < l$$

where

$$G_l(\alpha, \alpha) = \int \left(\frac{\alpha(r)}{a^0(r)} + 4\pi\gamma\Delta_l^{-1}\alpha \right) \alpha(r) dr, \quad (13b)$$

$$\Delta_l = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right) - \frac{l(l+1)}{r^2}.$$

For $l > 1$, $\Delta_l^{-1} > \Delta_1^{-1}$, and hence $G_l(\alpha, \alpha) > G_1(\alpha, \alpha)$. We now note that the operator $1/a^0(r) + 4\pi\gamma\Delta_1^{-1} \geq 0$, since $-\Delta_1 - 4\pi\gamma a^0(r) \geq 0$ (see the remark in p. 4). Consequently, $G_1(\alpha, \alpha) \geq 0$, and therefore $G(\alpha, \alpha) \geq 0$.

p. 6. Reduction to radial perturbations.

Theorem. If the solution $f^0(x, v) = h(\varepsilon^0(x, v))$, with $h'(\varepsilon) < 0$, is stable with respect to radial perturbations, then it is stable with respect to arbitrary perturbations*. In other words: if $K(\bar{\psi}, \bar{\psi}) \geq 0$ for any radial $\bar{\psi}$, then $K(\psi, \psi) \geq 0$ for any ψ .

Proof. Write the expansion $\psi = \bar{\psi} + \tilde{\psi}$ (see (11)). It is easy to verify that $K(\psi, \psi) = K(\bar{\psi}, \bar{\psi}) + K(\tilde{\psi}, \tilde{\psi})$. Every function $\tilde{\psi}$ satisfies the conditions of the lemma, since $\int \tilde{\psi} \mu(r) dx dv = 0$ for any $\mu(r)$. Consequently, $K(\tilde{\psi}, \tilde{\psi}) \geq 0$. The theorem is proved.

§ 2. Nonlinear consideration. A nonlinear treatment of the problem seems desirable, since in neutral problems (those without a distinguished direction of time) a linear treatment cannot give a final answer to the question of stability**.

p. 7. Variational principle. Consider Hamiltonian variations of the original function

$$f^0 \rightarrow f_\tau = f^0(x(\tau), v(\tau)); \quad dx/d\tau = \partial H/\partial v; \quad dv/d\tau = -\partial H/\partial x; \quad (14)$$

$H(x, v, \tau)$ is an arbitrary function of 7 variables. Let $E[f]$ be the total energy of the system,

$$E[f] = \int \frac{|v|^2}{2} f(x, v) dx dv - \frac{\gamma}{2} \int \frac{f(1)f(2)}{|x_1 - x_2|} d1 d2. \quad (15)$$

We shall show that $E[f]$ has a conditional extremum at f^0 , and find the conditions under which this extremum is a minimum. For small τ ,

$$f_\tau = f^0 + \tau \delta f + \frac{1}{2} \tau^2 \delta^2 f + o(\tau^2),$$

$$\delta f = \hat{H}_0[f^0]; \quad \delta^2 f = \hat{H}_0^2[f^0] - \hat{H}_1[f^0]; \quad H_0(x, v, \tau) = H(x, v, 0); \quad (16)$$

$$H_1(x, v, 0) = \frac{\partial}{\partial \tau} H(x, v, 0); \quad \hat{H}[f] = \frac{\partial H}{\partial v} \frac{\partial f}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial f}{\partial v}.$$

* This assertion is analogous to the following theorem, proved recently by one of the authors: for the stability of a gas sphere in its own gravitational field it is necessary and sufficient that: 1) it be stable with respect to radial perturbations; 2) the condition of absence of convection hold.

** In addition, in the linear treatment of this problem we were forced to restrict ourselves to perturbations $g = \delta f$ concentrated where $f^0(x, v) \neq 0$. This restriction is in no way required by the substance of the problem.

Substituting these expressions into (15), we obtain

$$E_\tau = E^0 + \tau \delta E + \frac{1}{2} \tau^2 \delta^2 E + o(\tau^2),$$

where

$$\begin{aligned} \delta E &= \int \left(\frac{|v|^2}{2} + \Phi^0(|x|) \right) \delta f \, dx \, dv = \int \varepsilon^0 \delta f \, dx \, dv, \\ \delta^2 E &= \int \varepsilon^0 \delta^2 f \, dx \, dv - \gamma \int \frac{\delta f(1) \delta f(2)}{|x_1 - x_2|} \, d1 \, d2. \end{aligned} \quad (17)$$

Using: a) the antisymmetry of the operators $\hat{H}_0$, $\hat{H}_1$; $\hat{A}$; b) $\hat{H}[h(\varepsilon)] = h'(\varepsilon) \hat{H}[\varepsilon]$; c) $\hat{A}[h(\varepsilon^0)] = \hat{A}[\varepsilon^0] = 0$, we obtain

$$\delta E = \int \varepsilon^0 h'(\varepsilon^0) \hat{H}[\varepsilon^0] \, dx \, dv = - \int \varepsilon^0 h'(\varepsilon^0) \hat{A}[H] \, dx \, dv = 0;$$

$$\delta^2 E = - \int h'(\varepsilon^0) b^2(x, v) \, dx \, dv - \gamma \int \frac{b(1)b(2)}{|x_1 - x_2|} h'(\varepsilon^0(1)) h'(\varepsilon^0(2)) \, d1 \, d2;$$

$$b(x, v) = \hat{H}_0[\varepsilon^0(x, v)].$$

It is easy to see that for nonnegativity of $\delta^2 E$ it is necessary that $h'(\varepsilon^0) \leq 0$. If $h'(\varepsilon^0) < 0$, then $\delta^2 E = \Gamma(\beta, \beta)$, where $\beta = \hat{H}[f^0] = -\hat{A}[h'(\varepsilon^0) H_0] = \hat{A}\psi$, and $\Gamma(\beta, \beta)$ is the quadratic form already known to us, (6). Thus: 1) for the Hamiltonian variations (14), $\delta E = 0$; 2) the conditions $\delta^2 E \geq 0$ for these variations coincide with the conditions of linear stability $h'(\varepsilon) < 0$; $K \geq 0^*$.

§ 8. Variational principle and stability. The set Q of all f_τ (for all possible $\hat{H}$)** is invariant with respect to equation (1). For ordinary differential equations, from the fact that a certain integral of the equations E has a minimum on the invariant surface Q at the (stationary) point f^0 , the stability of f^0 automatically follows (cf. with (4)). A complete proof of the stability of f^0 in our case might consist (as in the finite-dimensional case) in constructing a Lyapunov function, i.e. such a functional $\Lambda[f]$ that $d\Lambda/dt = 0$ by virtue of (1) and $\Lambda[f^0] = \min$. In view of the neutrality of our problem, it must be $d\Lambda/dt = 0$, i.e. Λ is an integral of equation (1). Besides the energy $E[f]$ (15), we know only the following integrals:

$$I[f] = \int J(f(x, v)) \, dx \, dv$$

($J(f)$ is an arbitrary function). We have not succeeded in constructing $\Lambda[f]$ from these integrals. It seems to us that this situation is typical for neutral infinite-dimensional problems: information about the first integrals is often insufficient for constructing a Lyapunov function, and one has to restrict oneself to a variational principle in the spirit of § 7 and finite-dimensional analogies. From this point of view, the problems considered in (4) (and the problem of a gravitating gas sphere), where such a construction is possible, are rather the exception than the rule.

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* The variational principle formulated is also valid for stationary solutions of a more general type.

** More precisely: Q consists of all $f(x, v) = f^0(X(x, v), V(x, v))$, where $x, v \rightarrow X, V$ is an arbitrary canonical transformation.

Note: Figure translations are in progress. See original paper for figures.

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