

ON THE SUMMABILITY (BOUNDEDNESS) OF SOLUTIONS OF ONE CLASS OF HYPOELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

S. V. Uspenskii

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.65196>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.9

MATHEMATICS

S. B. USPENSKII

ON THE SUMMABILITY (BOUNDEDNESS) OF SOLUTIONS OF ONE CLASS OF HYPOELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

(Presented by Academician S. L. Sobolev on 30 XII 1968)

In the present communication we continue the investigation, begun by the author in (1), of solutions of the equation

$$L(\partial/\partial x)U = f, \quad (1)$$

We shall assume below that the characteristic polynomial $L(is)$, with complex constant coefficients, has no real zeros for $|s| \neq 0$ and is quasi-homogeneous, i.e., for every $\lambda > 0$ the equality

$$P(is\lambda) = \lambda P(is)$$

holds.

The vector $\alpha = (a_1, \dots, a_n)$ will be called the homogeneity exponent of the operator. If $f \in L_p(C_\varepsilon^l)$, then, as was shown in ⁽¹⁾, equation (1) is always solvable in classes of functions that consist of summable (bounded) derivatives of order ρ , where $\rho \cdot \alpha = 1$. Solvability of equation (1) in narrower classes of functions containing summable (bounded) derivatives of lower orders, generally speaking, does not occur. However, one can distinguish a class of regular solutions for which, under certain conditions, one can guarantee the summability (boundedness) of derivatives of all orders ρ , where $0 \leq \rho \leq 1$. These conditions are formulated below in the form of four theorems.

This work is close in subject matter to the works ^(11, 2-4), where other conditions are imposed on the operator $L(\partial/\partial x)$, the most essential of which is the absence of real zeros of $L(is)$ in ⁽²⁾, and of common zeros of $L(is)$ and $\text{grad } L(is)$ in ^(3, 4). The results obtained also develop the well-known works of S. M. Nikol'skii ⁽⁵⁾, V. P. Il' in ⁽⁶⁾ on estimates of partial derivatives of functions of many variables, and the related works ^(7, 8) on coercivity.

Let us introduce definitions. Denote by R_n the Euclidean space of points

$$x = (x_1 \dots x_n); \quad x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}; \quad |x| = \left[\sum_{i=1}^n x_i^2 \right]^{1/2};$$

$$D^\beta = D^{\beta_1} \dots D^{\beta_n} = \partial^\beta / \partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}; \quad |l| = \sum_{i=1}^n l_i; \quad l \cdot \alpha = \sum_{i=1}^n l_i \alpha_i;$$

$$\Delta(f, hl_i) = f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i + h, \dots, x_n),$$

$$R_v[f(x)] = \prod_{i=1}^n v_i^{-1} \int_0^{v_1} \dots \int_0^{v_n} f(x + u) du; \quad (2)$$

$$\|f, L_p^l\| = \sum_{i=1}^n \|D^i f, L_p\|, \quad 1 < p < \infty; \quad (3)$$

$$\|f, L_p^{l, \alpha}\| = \sum_{\rho \cdot \alpha = 1} \|D^\rho f, L_p\| + \|D^\theta f, L_p^l\|; \quad (4)$$

$$\|f, L_{1, \gamma}\| = \int_{R_\nu} \prod_{i=1}^\nu (1 + |x_i|)^{\gamma_i} |f(x_1, \dots, x_n)| dR_\nu, \quad \gamma = (\gamma_1, \dots, \gamma_\nu),$$

$$1 \leq \nu \leq n; \quad (5)$$

$$\|f, C_\varepsilon, L_{1, \gamma}\| = \sup_{|v| \geq 0, x \in R_n} |v^\varepsilon R_\nu[\|f, L_{1, \gamma}\|]|; \quad (6)$$

$$\|f, C_\varepsilon\| = \sup_{|v| \geq 0, x \in R_{n-\nu}} |v^\varepsilon R_\nu[f(x)]|; \quad (7)$$

$$\varepsilon = (\varepsilon_{\nu+1}, \dots, \varepsilon_n), \quad \gamma = (\gamma_1, \dots, \gamma_\nu), \quad 0 \leq \nu \leq n-1.$$

For $p = \infty$, l an integer, $0 < \mu < 1$, put

$$\|f, C^{l+\mu}\| = \sum_{i=1}^n \left(\sup_{|h| \leq 1, x \in R_n} \left| \frac{\Delta(f_i^{(l_i)}, hl_i)}{h^{\mu_i}} \right| + \sum_{k=0}^{l_i} |D^k f| \right); \quad (8)$$

$$\|f, C_\varepsilon^{l+\mu}\| = \sum_{i=1}^n \|f, C_\varepsilon^{l+\mu}\| + \sum_{k=0}^{l_i} \|f^k, C_\varepsilon\|; \quad (9)$$

$$\|f, C_\varepsilon^{l+\mu, \alpha}\| = \sum_{\rho \cdot \alpha = 1} \|D^\rho f, C_\varepsilon^{l+\mu}\|. \quad (10)$$

We shall say that a function f belongs to the functional space $L_p^l(C_\varepsilon^l)$ if f belongs to the closure, in the corresponding norm, of smooth functions all of whose derivatives are summable (for which the norm is finite). Define a generalized solution of equation (1) as the limit, in the norm (4) (respectively, (10)), of solutions of the equation

$$L(\partial/\partial x)U_\nu = f_\nu,$$

where f_ν converges to f in the norm $L_p^l(C_\varepsilon^l)$.

We shall call a solution U of equation (1) regular if on every compact $G \in R_n$, $U \in W_p^{1/\alpha}(G)$ ($C^{1/\alpha+\mu}$), and there exists a natural number N such that for almost all $x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ one has

$$\lim_{|x_i| \rightarrow \infty} (1 + |x_i|)^N |D^\rho U| = 0 \quad (11)$$

for $i = 1, \dots, n$, $0 \leq \rho \cdot \alpha \leq 1$. Put

$$G_0(x) = \prod_{i=1}^n G_i(x_i) = \prod_{i=1}^n \exp[-(m_i \alpha_i)^{-1} x_i^{m_i}], \quad (12)$$

where the m_i are sufficiently large numbers,

$$G(x) = G_0(x) \sum_{i=1}^n x_i^{m_i} [L(ix)]^{-1}. \quad (13)$$

Then the following theorem on the representation of functions holds, generalizing the corresponding Radon formula (9).

Theorem 1. Let U be summable on every compact $G \subset R_n$, and let

$$\lim_{|v| \rightarrow \infty} R_\nu[U] = 0$$

for almost all $x \in R_n$.

Then for almost all $x \in R_n$,

$$U = \lim_{h \rightarrow 0} U_h = \lim_{h \rightarrow 0} \frac{L(\partial/\partial x)}{(2\pi)^n} \int_h^{1/h} v^{-|\alpha|} dv \int_{R_n} \widehat{G}\left(\frac{t-x}{v^\alpha}\right) U(t) dt. \quad (14)$$

Corollary. Let $f \in L_p(C_\varepsilon^l)$, and let U be a regular solution of (1) satisfying the condition

$$\lim_{|v| \rightarrow \infty} R_\nu[D^\rho U] = 0 \quad \text{for almost all } x \in R_n \text{ (for all } x). \quad (15)$$

Then for almost all $x \in R_n$ (for all x) one has

$$D^\rho U = \lim_{h \rightarrow 0} \frac{1}{(2\pi)^h} \int_h^{1/h} v^{-|\alpha|-\rho\alpha} dv \int_{R_n} D^\rho \widehat{G}\left(\frac{t-x}{v^\alpha}\right) f(t) dt. \quad (16)$$

Using Theorem 1, one can obtain the following assertions.

Theorem 2. Let $f \in C_\varepsilon^{l+\mu, \alpha}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $0 < \varepsilon_i \leq 1$, $\rho = (\rho_1, \dots, \rho_n)$, $\rho_i \geq 0$, $i = 1, \dots, n$, $0 < \mu_i < 1$, $l_i > 0$, $\mu_i = \mu\alpha_i^{-1}$. If $(\rho + \varepsilon)\alpha > 1$, $\rho\alpha < 1$, then there exists a solution of equation (1) $U \in C_\varepsilon^{l+\mu, \alpha}$, whose partial derivative $D^\rho U \in C$, and the estimate holds

$$C_1 \|f, C_\varepsilon^{l+\mu}\| \leq \|U, C_\varepsilon^{l+\mu, \alpha}\| + \|D^\rho U, C_\varepsilon^{l+\mu}\| \leq C_2 \|f, C_\varepsilon^{l+\mu}\|, \quad (17)$$

where $\bar{\varepsilon}_i < \varepsilon_i(1 - (1 - \rho\alpha)/|\alpha|)$, $i = 1, \dots, n$. Every regular solution v satisfying condition (15) has estimate (17).

Corollary. Let $f \in L_p(R_n) \cap C^{l+\mu}(R_n)$, $1 < p < \infty$, $l_i > 0$, $0 < \mu_i < 1$; if $|\alpha| > p$, $\mu_i = \mu\alpha_i^{-1}$, then equation (1) has a solution

$$U_0 \in C_\varepsilon^{l+\mu+1/\alpha}, \quad C_1 \|f, C_\varepsilon^{l+\mu}\| \leq \|U_0, C_\varepsilon^{l+\mu+1/\alpha}\| \leq C_2 (\|f, L_p\| + \|f, C^{l+\mu}\|),$$

where

$$\bar{\varepsilon}_i < \frac{1}{p} \left(\frac{|\alpha| - 1}{|\alpha|} \right), \quad i = 1, 2, \dots, n.$$

Every regular solution v satisfying condition (15) for $|\rho| = 0$ coincides with U_0 .

The following assertion generalizes to the case of operators the known Sobolev embedding theorem ⁽¹⁰⁾ for the limiting exponent (see also ^(5,6)).

Theorem 3. Let $f \in L_p(R_n)$, $1 < p < \infty$, $p < q < \infty$, $\varkappa = \rho\alpha + (1/p - 1/q)|\alpha| = 1$.

Then there exists a solution U_0 of equation (1), for which the estimate holds

$$C_1 \|f, L_p\| \leq \|U_0, L_p^{l,\alpha}\| + \|D^\rho U_0, L_q\| \leq C_2 \|f, L_p\|. \quad (18)$$

Every regular solution v , for which (15) holds, satisfies estimate (18).*

Remark. From the embedding theorem for the limiting exponent it follows that Theorem 2 is sharp in the limiting sense.

Theorem 4. Let $f \in L_p \cap L_r$, $1 < p \leq r < q < \infty$ and, moreover, $\varkappa_1 = \rho\alpha + (1/p - 1/q)|\alpha| \geq 1$, $\varkappa_2 = \rho\alpha + (1/r - 1/q)|\alpha| \leq 1$.

Then equation (1) has a solution U_0 , for which the estimate holds

$$C_1 (\|f, L_p\| + \|f, L_r\|) \leq \|U_0, L_p^{l,\alpha}\| + \|U_0, L_r^{l,\alpha}\| + \|D^\rho U_0, L_q\| \leq C_2 (\|f, L_p\| + \|f, L_r\|). \quad (19)$$

For every regular solution v satisfying (15), estimate (19) is valid.

Theorem 2 can be strengthened if certain structural requirements are imposed on the right-hand side of equation (1) (oddness, equality to zero of certain integrals, etc.). The following Theorem 5 gives some idea of the character of the behavior of solutions in this case.

Theorem 5. Let $f = C^{l+\mu}$, and let the norms be finite

$$\|f, C_\varepsilon, L_{1,\gamma}\| < \infty, \quad \|f, C_{\hat{\varepsilon}}\| < \infty,$$

$$\varepsilon = (\varepsilon_{\nu+1}, \dots, \varepsilon_n), \quad \gamma = (\gamma_1, \dots, \gamma_\nu), \quad \hat{\varepsilon} = (\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n),$$

$$0 < \varepsilon_i \leq 1, \quad \gamma_i > 1, \quad 0 < \hat{\varepsilon}_i \leq 1, \quad i = 1, 2, \dots, n,$$

and, moreover, for each i , $1 \leq i \leq \nu$, it holds that

$$f(x_1, \dots, -x_i, \dots, x_n) = -f(x_1, \dots, x_i, \dots, x_n).$$

* Coercive estimates of the form (18) were obtained earlier by O. V. Besov (8).

Then, if

$$\varkappa_1 = \rho \cdot \alpha + \sum_{i=\nu+1}^n \varepsilon_i a_i + 2 \sum_{i=1}^n a_i > 1, \quad \varkappa_2 = (\rho + \hat{\varepsilon})\alpha < 1,$$

then there exists a solution U_0 of equation (1) for which

$$\begin{aligned} C_1 \|f, C_{\bar{\varepsilon}}^{l+\mu}\| &\leq \|U_0, C_{\bar{\varepsilon}}^{l+\mu, \alpha}\| + \|D^{\rho} U_0, C\| \leq \\ &\leq C_2 (\|f, C_{\bar{\varepsilon}} L_{1, \gamma}\| + \|f, C_{\varepsilon}\| + \|f, C^{l+\mu}\|), \end{aligned} \quad (20)$$

$$\bar{\varepsilon}_i < \begin{cases} 1, & i = 1, \dots, \nu, \\ \varepsilon_i, & i = \nu + 1, \dots, n. \end{cases}$$

For any regular solution v of equation (1) satisfying (15), the estimate (20) holds.

Corollary. Let $f \in C^{l+\mu}$, $\|f, L_{1, \gamma}\| < \infty$, $\|f, C_{\varepsilon}\| < \infty$, $\gamma = (\gamma_1, \dots, \gamma_n)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\gamma_i > 1$, $0 < \varepsilon_i \leq 1$, $i = 1, 2, \dots, n$,

$$\kappa_1 = |\alpha| + \rho\alpha + \min_{1 \leq i \leq n} a_i > 1, \quad \kappa_2 = (\varepsilon + \rho)\alpha < 1.$$

Then, if $\int_{R_n} f dR_n = 0$, there exists a solution U_0 of equation (1) satisfying the estimate

$$\begin{aligned} C_2 \|f, C_{\bar{\varepsilon}}^{l+\mu}\| &\leq \|U_0, C_{\bar{\varepsilon}}^{l+\mu, \alpha}\| + \|D^{\rho} U_0, C\| \leq \\ &\leq C_1 (\|f, L_{1, \gamma}\| + \|f, C^{l+\mu}\| + \|f, C_{\varepsilon}\|), \quad \bar{\varepsilon}_i < 1, \quad i = 1, 2, \dots, n. \end{aligned} \quad (21)$$

Every regular solution v of equation (1) satisfies condition (21), if condition (15) is fulfilled.

Institute of Mathematics Siberian Branch of the Academy of Sciences of the USSR Novosibirsk

Received 23 XII 1968

CITED LITERATURE

- ¹ S. V. Uspenskii, DAN, **181**, No. 3, 562 (1968).*
- ² V. P. Palamodov, DAN, **129**, No. 4, 740 (1959).
- ³ B. R. Vainberg, UMN, **21**, issue 3 (129), 115 (1966).
- ⁴ V. V. Grushin, Mat. sborn., **61** (103), 2, 147 (1963).
- ⁵ S. M. Nikol'skii, Mat. sborn., **61** (103), 2, 224 (1963).
- ⁶ V. P. Il'in, Tr. Mat. inst. im. V. A. Steklova AN SSSR, **84**, 144 (1965).
- ⁷ K. T. Smith, Bull. Am. Math. Soc., **67**, 368 (1961).
- ⁸ O. V. Besov, Mat. sborn., **73** (115), 4, 585 (1967).

⁹ F. John, *Plane Waves and Spherical Means*, IL, 1968.

¹⁰ S. L. Sobolev, *Mat. sborn.*, **4**, No. 3, 471 (1938).

¹¹ V. P. Palamodov, *DAN*, **122**, No. 3, 528 (1960).

* In the author' s work (1), an inaccuracy was allowed in the formulation of Theorem 4. In the condition of the theorem the following additional conditions should be introduced: $l = (l_1, \dots, l_n)$, $l_i = \bar{l}_i + \mu_i$, $\bar{l}_i$ an integer, $0 < \mu_i < 1$, $\mu_i = \mu a_i^{-1}$, where $a = (a_1 \dots a_n)$ is the homogeneity exponent of the operator $L(D)$.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.