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Abstract

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MECHANICS

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VARIATIONAL PRINCIPLE AND EQUATIONS OF STATE FOR CONTINUOUS MEDIA*

(Presented by Academician L. I. Sedov, 4 VII 1968)

Below, models of material continuous media are considered within the framework of the special theory of relativity on the basis of a variational principle of the form ⁽¹⁻³⁾

$$\delta \int_{V_4} \Lambda d\tau + \delta W^* + \delta W = 0. \quad (1)$$

In the variational principle (1), as admissible extremals we shall consider piecewise smooth functions that may have a finite discontinuity on certain three-dimensional surfaces. The variation is applied to the defining parameters of the medium $\mu^A(\xi^p)$, which may be components of tensors or spinors ⁽⁴⁻⁶⁾, and to the law of motion of the medium $\delta x^i = \tilde{x}^i(\xi^p) - x^i(\xi^p)$ (ξ^p are the variables of the accompanying coordinate system for the medium; the functions $x^i(\xi^p)$ determine the law of motion of the points of the medium). The total variation of the defining parameters $\delta_1 \mu^A$ is given by the equality $\delta_1 \mu^A = \partial \mu^A + \delta x^i \partial_i \mu^A$, where $\partial_i = \partial / \partial x^i$, $\partial \mu^A = \mu^A(x^p) - \tilde{\mu}^A(x^p)$ is the variation of the form of the functions $\mu^A(x^p)$ at fixed x^p . Variations of the defining parameters in the volume V_4 are regarded as independent arbitrary functions belonging to the same class of functions as the admissible extremals. The relation between the variations of the defining parameters on the sides ($\pm$) of the discontinuity surface S is established by the choice of the class of functions for the admissible extremals.

1. Equations of state for continuous media. Consider the case when Λ is defined as a function of m derivatives with respect to the coordinates of the parameters μ^A and of the quantities $x_p^i = \partial x^i / \partial \xi^p$. We choose the functional δW^* in the form

$$\delta W^* \int_{V_4} (P_i^* \delta x^i + M_A^* \delta \mu^A) d\tau - \int_{\Sigma+S_{\pm}} \left(\sum_{\nu=0}^m P_i^{*kj_1 \dots j_\nu} \partial_{t_1 \dots j_\nu} \delta x^i + \sum_{\nu=0}^{m-1} M_A^{*kj_1 \dots j_\nu} \partial_{j_1 \dots j_\nu} \delta \mu^A \right) n_k d\sigma, \quad (2)$$

where Σ is the surface bounding the volume V_4 ; S is the part of the discontinuity surface of the defining parameters lying in V_4 . The sign $S_{\pm}$ indicates that the integration is carried out over the two sides of the surface S . The tensors P^*, M^* are assumed to be prescribed functions of the parameters of the medium. n_k are the components of the unit (imaginary-unit) vector of the outward normal to the surfaces Σ, S_+, S_- ; $d\sigma$ is an element of the surfaces Σ, S ,

$$\partial_{j_1 \dots j_\nu} = \partial_{j_1} \dots \partial_{j_\nu}.$$

* The main results of the present work were presented at the Third All-Union Congress on Theoretical and Applied Mechanics, January 1968.

Using the usual methods (7), from (1) we determine the functional δW

$$\delta W = \int_{\Sigma+S_{\pm}} \left(\sum_{\nu=0}^m P_i^{kj_1 \dots j_\nu} \partial_{j_1 \dots j_\nu} \delta x^i + \sum_{\nu=0}^{m-1} M_A^{kj_1 \dots j_\nu} \partial_{j_1 \dots j_\nu} \delta \mu^A \right) n_k d\sigma. \quad (3)$$

Here the abbreviated notation has been introduced

$$\frac{\delta \Lambda}{\delta \partial_{j_1 \dots j_\nu} \mu^A} = \sum_{\gamma=0}^{m-\nu} (-1)^\gamma \partial_{i_1 \dots i_\gamma} \frac{\partial \Lambda}{\partial_{i_1 \dots i_\gamma j_1 \dots j_\nu} \mu^A},$$

$$M_A^{kj_1 \dots j_\nu} = {}^* M_A^{kj_1 \dots j_\nu} - \frac{\partial \Lambda}{\delta \partial_{kj_1 \dots j_\nu} \mu^A},$$

$$P_i^k = {}^* P_i^k + \sum_{\gamma=0}^{m-1} \left(\frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_\gamma} \mu^A} \partial_{j_1 \dots j_\gamma i} \mu^A + \frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_\gamma} x_p^s} \partial_{j_1 \dots j_\gamma i} x_p^s \right) - \Lambda \delta_i^k,$$

$$P_i^{kj_1 \dots j_\nu} = {}^* P_i^{kj_1 \dots j_\nu} - \sum_{\gamma=\nu-1}^{m-1} C_\gamma^{\nu-1} \frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_{\nu-1} s_\nu \dots s_\gamma x_p^i}} \partial_{s_\nu \dots s_\gamma} x_p^{j_\nu} + \sum_{\gamma=\nu}^{m-1} C_\gamma^\nu \left(\frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_\gamma} \mu^A} \partial_{j_{\nu+1} \dots j_\gamma i} \mu^A + \frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_\gamma} x_p^s} \partial_{j_{\nu+1} \dots j_\gamma i} x_p^s \right), \quad 0 < \nu < m, \quad (4)$$

$$P_i^{kj_1 \dots j_m} = {}^* P_i^{kj_1 \dots j_m} - \frac{\delta \Lambda}{\delta \partial_{kj_1 \dots j_{m-1} x_p^i}} x_p^{j_m}.$$

C_γ^ν is the number of combinations of γ taken ν at a time. It is important to note that the tensors P, M , defined by formulas (4) in a certain concrete form, are in fact determined by the variational principle nonuniquely, since the

quantities δx^i , $\delta \mu^A$ and their derivatives are dependent on the surfaces Σ, S . It is easy to see that, without changing the variational equality (1), one can redefine the equations of state (4) by adding to the right-hand side of equality (3) some functional $\delta \bar{W}$ that is identically zero or zero by virtue of the Euler equations. In particular, the functional $\delta \bar{W}$, identically equal to zero, may be defined by a formula of the form $\int \partial_i \partial_j \Omega^{ij} d\tau$, where $\Omega^{ij} = -\Omega^{ji}$ are arbitrary piecewise smooth functions, linear with respect to the variations $\delta x^i, \delta \mu^A$ and their gradients. It is easy to see that the concrete determination of the tensors P, M depends on the adopted technique of variation. It is clear that, owing to the "subjective" arbitrariness in the definition of the tensors P, M , it is not appropriate to attach to them a physical meaning.*

We shall assume that the piecewise smooth surfaces Σ, S admit a parametric representation $x^i = x^i(u^\alpha)$, and let $G^{\alpha\beta}$ be the contravariant components of the first metric tensor of the surfaces Σ, S . Using the obvious formula

$$\frac{\partial^\nu}{\partial n^\nu} \partial_j \chi^A = n^{j_1 \dots j_\nu} \xi_j^\beta {}^* \nabla_\beta \partial_{j_1 \dots j_\nu} \chi^A + (n^p n_p) n_j \frac{\partial^{\nu+1}}{\partial n^{\nu+1}} \chi^A, \quad n^{j_1 \dots j_\nu} = n^{j_1} \dots n^{j_\nu},$$

where $\partial^\nu / \partial n^\nu = n^{j_1 \dots j_\nu} \partial_{j_1 \dots j_\nu}$; $\xi_j^\beta = G^{\alpha\beta} g_{ij} \partial x^i / \partial u^\alpha$; ${}^* \nabla_\alpha$ is the symbol of covariant differentiation on the surface, one can obtain the recurrence relation

$$\begin{aligned} \partial_{j_1 \dots j_{\nu-1} j_\nu} \chi^A &= \xi_{j_1 \dots j_{\nu-1} j_\nu}^{i_1 \dots i_{\nu-1} \alpha} {}^* \nabla_\alpha \partial_{i_1 \dots i_{\nu-1}} \chi^A + (-1)^\nu n_{j_1 \dots j_{\nu-1} j_\nu} \frac{\partial^\nu}{\partial n^\nu} \chi^A, \\ \xi_{j_1 \dots j_{\nu-1} j_\nu}^{i_1 \dots i_{\nu-1} \alpha} &= \left[\delta_{j_1}^k \delta_{j_2}^{i_1} \dots \delta_{j_\nu}^{i_{\nu-1}} + \sum_{\gamma=2}^{\nu-1} (n^p n_p)^{\gamma-1} n^{i_1 \dots i_{\gamma-1}} n_{j_1 \dots j_{\gamma-1}} \delta_{j_\gamma}^k \delta_{j_{\gamma+1}}^{i_\gamma} \dots \delta_{j_\nu}^{i_{\nu-1}} \right. \\ &\quad \left. + (n^p n_p)^{\nu-1} n^{i_1 \dots i_{\nu-1}} n_{j_1 \dots j_{\nu-1}} \delta_{j_\nu}^k \right] \xi_k^\alpha. \end{aligned} \quad (5)$$

* In this connection we note that in the known theories the determination of the tensors of energy-momentum, angular momentum, currents, and other observed quantities is associated with the tensors P, M . In particular, the moment theory of elasticity is based on the physical interpretation of the tensors P .

Taking (5) into account, we obtain

$$\begin{aligned}
M_A^{j_1 \dots j_\nu} \partial_{j_1 \dots j_\nu} \chi^A &= (n^p n_p)^\nu M_A^{j_1 \dots j_\nu} n_{j_1 \dots j_\nu} \frac{\partial^\nu}{\partial n^\nu} \chi^A + \chi^A \nabla_\alpha^* H_{A(\nu)}^\alpha \\
&\quad - \nabla_\alpha^* (H_{A(\nu)}^\alpha \chi^A) + \sum_{\gamma=1}^{\nu-1} (n^p n_p)^\gamma n_{j_1 \dots j_\gamma} \nabla_\alpha^* H_{A(\nu)}^{j_1 \dots j_\gamma \alpha} \frac{\partial^\gamma}{\partial n^\gamma} \chi^A \\
&\quad - \sum_{\gamma=1}^{\nu-1} \nabla_\alpha^* (H_{A(\nu)}^{j_1 \dots j_\gamma \alpha} \partial_{j_1 \dots j_\gamma} \chi^A).
\end{aligned} \tag{6}$$

Here the notation is

$$\begin{aligned}
H_{A(\nu)}^{j_1 \dots j_\gamma \alpha} &= (-1)^{\nu-\gamma} \xi_{r_1 \dots r_\gamma r_{\gamma+1}}^{j_1 \dots j_\gamma \alpha} \nabla_\beta^* \left\{ \xi_{l_1 \dots l_{\gamma+1} l_{\gamma+2}}^{r_1 \dots r_{\gamma+1} \beta} \nabla_\varepsilon^* \left[\xi_{q_1 \dots q_{\gamma+2} q_{\gamma+3}}^{l_1 \dots l_{\gamma+2} \varepsilon} \nabla_\rho^* \right. \right. \\
&\quad \left. \left. \times [\dots \xi_{i_1 \dots i_{\nu-2} i_{\nu-1}}^{p_1 \dots p_{\nu-2} \mu} \nabla_\sigma^* (\xi_{m_1 \dots m_{\nu-1} m_\nu}^{i_1 \dots i_{\nu-1} \sigma} M_A^{m_1 \dots m_{\nu-1} m_\nu})] \right] \right\}.
\end{aligned} \tag{7}$$

From the definition it is clear that the tensors H satisfy the recurrence relation

$$H_{A(\nu)}^{i_1 \dots i_{\gamma-1} \alpha} = -\xi_{j_1 \dots j_{\gamma-1} j_\gamma}^{i_1 \dots i_{\gamma-1} \alpha} \nabla_\beta^* H_{A(\nu)}^{j_1 \dots j_\gamma \beta}. \tag{8}$$

Replacing the integrands of the functional δW by formula (6), we obtain the expression of this functional in the following canonical form

$$\delta W = \int_{\Sigma + S_\pm} \left(\sum_{\nu=0}^m \mathcal{P}_{i(\nu)} \frac{\partial^\nu \delta x^i}{\partial n^\nu} + \sum_{\nu=0}^{m-1} \mathcal{M}_{A(\nu)} \frac{\partial^\nu \delta \mu^A}{\partial n^\nu} \right) d\sigma - \int_{\Sigma + S_\pm} \nabla_\alpha^* \left(\sum_{\nu=0}^{m-1} \mathcal{P}_i^{j_1 \dots j_\nu \alpha} \partial_{j_1 \dots j_\nu} \delta x^i + \sum_{\nu=0}^{m-2} \mathcal{M}_A^{j_1 \dots j_\nu} \partial_{j_1 \dots j_\nu} \right)$$

where

$$\mathcal{M}_A^\alpha = \sum_{\gamma=1}^m H_{A(\gamma)}^\alpha, \quad \mathcal{M}_A^{j_1 \dots j_\nu \alpha} = \sum_{\gamma=1}^{m-\nu} H_{A(\gamma+\nu)}^{j_1 \dots j_\nu \alpha},$$

$$\mathcal{M}_{A(\nu)} = (n^p n_p)^\nu \left(M_A^{k j_1 \dots j_\nu} n_k + \nabla_\alpha^* \mathcal{M}_A^{j_1 \dots j_\nu \alpha} \right) n_{j_1 \dots j_\nu}. \tag{10}$$

The tensors $\mathcal{P}$ in formula (9) are obtained from formulas (10) by replacing in them the symbols $M, \mathcal{M}$ by the symbols $P, \mathcal{P}$. In the case where the integrands in the second integral in (9) are continuous on the surfaces $\Sigma + S_\pm$, the second integral in (9) vanishes.

From formulas (7), (8) it follows that the tensors $\mathcal{M}$ satisfy the recurrence relations

$$\mathcal{M}_A^{j_1 \dots j_{m-1} \alpha} = -\xi_{j_1 \dots j_{m-1} j_m}^{i_1 \dots i_{m-1} \alpha} M_A^{k j_1 \dots j_m} n_k,$$

$$\mathcal{M}_A^{i_1 \dots i_{\nu-1} \alpha} = -\xi_{j_1 \dots j_{\nu-1} j_\nu}^{i_1 \dots i_{\nu-1} \alpha} \left(M_A^{k j_1 \dots j_\nu} n_k + \nabla_\beta^* \mathcal{M}_A^{j_1 \dots j_\nu \beta} \right). \quad (11)$$

Formulas (11) are considerably simpler than formulas (10), and for the computation of the tensors $\mathcal{P}, \mathcal{M}$ it is more convenient to use the recurrence formulas (11) instead of the direct formulas (10).

Since on the surfaces Σ, S the quantities $\delta x^i, \delta \mu^A$ and the normal derivatives of these quantities are independent, it is obvious that the components $\mathcal{P}_{i(\nu)}, \mathcal{M}_{A(\nu)}$ are determined by the variational principle in a unique way.

2. Conditions on discontinuity surfaces

If the surface S is sufficiently smooth and the parameters of the medium are continuous on both sides of the surface S , then the variational conditions on the discontinuity surface may be written in the form

$$\int_{S_\pm} \left(\sum_{\nu=0}^m \mathcal{P}_{i(\nu)} \frac{\partial^\nu \delta x^i}{\partial n^\nu} + \sum_{\nu=0}^{m-1} \mathcal{M}_{A(\nu)} \frac{\partial^\nu \delta \mu^A}{\partial n^\nu} \right) d\sigma = 0. \quad (12)$$

Consider the case in which the first r derivatives $\partial_{1\dots r} \mu^A$ of the parameters μ^A and the first s derivatives $\partial_{j_1 \dots j_{s-1}} x_{j_s}^i$ from the law of motion are continuous on the surface S , while derivatives of the following orders undergo an arbitrary finite discontinuity on S . For definiteness, set $0 < r < s$. From the assumption on the class of functions under consideration it follows that the quantities $\partial^\nu \delta x^i / \partial n^\nu$, $\partial^\mu \delta \mu^A / \partial n^\mu$ for $\nu \leq s$, $\mu \leq r$ are continuous, while for $\nu > s$, $\mu > r$ they are independent on the surface S . Using the arbitrariness of S and the indicated properties of the variations of the parameters on S , from (12) we obtain the system of differential conditions on the surface of discontinuity

$$[\mathcal{P}_{i(\nu)}]_S = 0 \quad (\nu \leq s), \quad \mathcal{P}_{i(\nu)}|_S = 0 \quad (\nu > s),$$

$$[\mathcal{M}_{A(\nu)}]_S = 0 \quad (\nu \leq r), \quad \mathcal{M}_{A(\nu)}|_S = 0 \quad (\nu > r) \quad (13)$$

(the square brackets $[\]_S$ denote the difference of the limiting values on S of the quantities enclosed in the brackets, and it is assumed that $n_i^+ = n_i^-$).

The functional δW , defined by formula (3), was obtained under the assumption that the surface S is not varied in the accompanying coordinate system. Assuming that, under variation of the surface of discontinuity $\xi^i = \xi^i(u^\alpha)$, the points u^α of the surface receive an increment of coordinates $d\xi^i = \bar{\xi}^i(u^\alpha) - \xi^i(u^\alpha)$, and

using (3), (13), we compute the value of the functional δW for zero variations of the defining parameters at the points u^α of the varied surface S and on the surface Σ :

$$\delta W = - \int_{S_\pm} \left(\Lambda \hat{n}_p + \mathcal{P}_{i(s)} \hat{n}^k n^{j_1 \dots j_{s-1}} x_p^l \partial_{j_1 \dots j_{s-1}} x_k^i + \mathcal{M}_{A(r)} n^{j_1 \dots j_r} x_s^l \partial_{j_1 \dots j_r} \mu^A \right) d\xi^p d\sigma, \quad (14)$$

where $\hat{n}_p = x_p^i n_p$. Taking into account the kinematic conditions for the functions $\partial_{j_1 \dots j_r} \mu^A$, $\partial_{j_1 \dots j_{s-1}} x_k^i$, from (14) we obtain an additional condition due to the variation of the surface of discontinuity,

$$\left[\Lambda + \mathcal{P}_{i(s)} \hat{n}^k \partial^s x_k^i / \partial n^s + \mathcal{M}_{A(r)} \partial^{r+1} \mu^A / \partial n^{r+1} \right]_S = 0. \quad (15)$$

We separately consider the case in which the quantities $\partial_i \mu^A$, x_p^i undergo a discontinuity on S (the case $m = s = 0$). In this case (14) passes into the equality

$$\delta W = - \int_{S_\pm} \left(\mathcal{P}_{i(0)} x_p^i + \mathcal{M}_{A(0)} x_p^i \partial_i \mu^A + \Lambda \hat{n}_p \right) d\xi^p d\sigma = 0. \quad (16)$$

Taking into account the kinematic conditions $[\hat{n}_k / \sqrt{-\hat{g}}]_S = 0$ ($\hat{g} = \det \hat{g}_{pq}$, $\hat{g}_{pq} = x_p^i x_q^j g_{ij}$), from (16) we obtain*

$$\left[\sqrt{-\hat{g}} \left(\Lambda + \mathcal{P}_{i(0)} n^i + \mathcal{M}_{A(0)} \partial \mu^A / \partial n \right) \right]_S^+ = 0. \quad (17)$$

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* Conditions on the surface of discontinuity within the framework of Newtonian mechanics for the case in which Λ depends on x_p^i , $\partial_k x_p^i$, were considered in work (8). The conditions obtained in that work may be considered satisfactory only in certain particular examples not specified in (8).

Note: Figure translations are in progress. See original paper for figures.

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