

DUALITY THEORY FOR TOPOLOGICAL VECTOR SPACES OVER A DISCRETE FIELD

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Abstract

Full Text

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MATHEMATICS

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DUALITY THEORY FOR TOPOLOGICAL VECTOR SPACES OVER A DISCRETE FIELD

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I. Definition 1. A linear space E over a field P , endowed with a topology u , is called a **topological vector group** ⁽¹⁾ (abbreviated t.v.g.) if the mappings

$$\xi_1 : E \times E \rightarrow E (\xi_1(x, y) = x + y), \quad \xi_2 : P \times E \rightarrow E (\xi_2(\lambda, x) = \lambda x)$$

are continuous when P is endowed with the discrete topology. E , endowed with the discrete topology, is a t.v.g. Every topological linear space is also a t.v.g. Linear topological spaces are also t.v. groups ⁽²⁾, § 10).

Everywhere in what follows it is assumed that P is either the field of real numbers R or the field of complex numbers C .

We shall consider only those t.v.g.'s which possess a base of neighborhoods of zero consisting of convex sets. Such t.v.g.'s are called **locally convex groups** (l.c.g.).

L.c.g.'s possess a base of neighborhoods of zero η , which satisfies the conditions: a) every $U \in \eta$ is absolutely convex; b) for any $\lambda \in P$, $\lambda \neq 0$, and $U \in \eta$, there exists $V \in \eta$ such that $V \subset \lambda U$.

For $P = R$ this assertion is obvious, and for $P = C$ it was proved in ⁽¹⁾. Conversely, every filter base η in E possessing properties a) and b) is a base of neighborhoods of zero for some topology relative to which E becomes an l.c.g.

The conjugate $E' \equiv (E, u)'$ to the l.c.g. (E, u) is the set of all continuous linear mappings $f : E \rightarrow P$, where P is endowed with the topology usual for the line or the plane.

Theorem 1 (Hahn–Banach). Let (E, u) be an l.c.g. and let f be a continuous functional defined on a linear subspace $E_1 \subset E$. Then there exists $\bar{f} \in E'$ such that $\bar{f}(x) = f(x)$ for $x \in E_1$.

Corollary 1. If (E, u) is a separated l.c.g. and $x_0 \in E$, $x_0 \neq 0$, then there exists $f_0 \in E'$ such that $f_0(x_0) = 1$.

By $\sigma(E, E')$ we shall denote the weakest topology in E with respect to which all $f \in E'$ are continuous. Let (E, u) be an l.c.g. and E_1 a linear subspace of E ; the topology u induces on E_1 some topology u_1 . Put $E'_1 = (E_1, u_1)'$.

Corollary 2. The topology $\sigma(E, E')$ induces on E_1 the topology $\sigma(E_1, E'_1)$.

Corollary 3. Let (E, u) be an l.c.g. and let E_1 be a closed linear subspace; then E_1 is $\sigma(E, E')$ -closed.

Proposition 1. Let U be an absolutely convex neighborhood of zero of the l.c.g. (E, u) . Then

$$\frac{1}{2}\overline{U}^{\sigma(E, E')} \subset U.$$

The meaning of this proposition is that every l.c.g. has a base of neighborhoods consisting of weakly closed absolutely convex sets.

Definition 2. Let (E, τ) be a separated l.c.g. A set $A \subset E$ is called **precompact** if $\bar{A}^\tau$ is complete in the topology τ . In other words, A is precompact if every τ -Cauchy net $\{x_\alpha\}$, $x_\alpha \in A$, converges in E .

Let (E, F) be a separated dual pair with bilinear function $\langle e, f \rangle$. For $A \subset E$ put

$$A^0 = \{f \in F : |\langle A, f \rangle| \leq 1\}.$$

Theorem 2 (V. L. Shmul'yan; see ⁽³⁾). Let (E, F) be a separated dual pair. An absolutely convex set $A \subset E$ is $\sigma(E, F)$ -precomplete if and only if every linear function s on F ($s \in F^*$), restricted to A^0 , is representable by an element of E (i.e. $s(f) = \langle e_s, f \rangle$ for some $e_s \in E$).

Theorem 3 (Banach—Alaoglu). Let (E, τ) be an l.v.s., and let U be a neighborhood of zero in (E, τ) . Then U^0 is a $\sigma(E', E)$ -complete set in E' .

Let (E, F) be a separated dual pair and let τ be an l.v.s. topology in E , consistent with the duality between E and F , i.e. $(E, \tau)'$ is isomorphic to F . Let $\{U_\gamma : \gamma \in \Gamma\}$ be a base of neighborhoods of zero for (E, τ) , consisting of $\sigma(E, F)$ -closed absolutely convex sets. Since $U_\gamma^{00} = U_\gamma$, by Theorem 3, τ is the topology of uniform convergence on the family $\{U_\gamma^0 : \gamma \in \Gamma\}$ of $\sigma(F, E)$ -complete absolutely convex subsets of F^* .

It turns out that the topology $\nu(E, F)$ of uniform convergence on all absolutely convex $\sigma(F, E)$ -complete subsets of F is also consistent with the duality between E and F .

Theorem 4. Let (E, F) be a separated dual pair. Every l.v.s. topology τ in E consistent with the duality between E and F satisfies the condition

$$\sigma(E, F) \leq \tau \leq \nu(E, F)$$

and is the topology of uniform convergence on some family of absolutely convex $\sigma(F, E)$ -complete subsets of F .

In this theorem the words “ $\sigma(E, F)$ -complete” may be replaced by the words “ $\sigma(F, E)$ -precomplete.”

Corollary 1. The absolutely convex hull $\Gamma(A_1 \cup A_2)$ of the union of two $\sigma(F, E)$ -precomplete absolutely convex sets A_1 and A_2 is a $\sigma(F, E)$ -precomplete set.

II. From Theorem 4 it follows

Theorem 5. Let $\varphi : E \rightarrow F$ be a continuous linear mapping of a separated l.v.s. E into a separated l.v.s. F , and let A be a $\sigma(E, E')$ -precomplete absolutely convex subset of E . Then φA is a $\sigma(F, F')$ -precomplete set and

$$\varphi A^{\sigma(F, F')} \subset \varphi E.$$

From Theorem 5 and Theorem 3 it follows

Proposition 2. A continuous linear mapping $\varphi : E \rightarrow F$ of a separated l.v.s. E into a separated l.v.s. F remains continuous if E and F are endowed with the topologies $\nu(E, E')$ and $\nu(F, F')$, respectively.

Corollary. The quotient group F of an l.v.s. E , endowed with the topology $\nu(E, E')$, is an l.v.s. with the topology $\nu(F, F')$.

III. Let E be a separated l.v.g. with adjoint E' . A set $A' \subset E'$ will be called, following V. Pták⁽⁵⁾, **almost closed** if $A' \cap U^0$ is $\sigma(E', E)$ -closed for every neighborhood U in E . An l.v.g. E will be called **perfectly complete** if every almost closed linear subspace in E' is $\sigma(E', E)$ -closed.

A linear mapping $\varphi : (E, u) \rightarrow (F, v)$ of a separated l.v.g. (E, u) into a separated l.v.g. (F, v) will be called **almost open** if $\varphi U^{\sigma(F, F')}$ is a neighborhood of zero in (F, v) whenever U is a neighborhood of zero in (E, u) ^{**}. Analogously one defines almost continuity of a mapping $\varphi : (E, u) \rightarrow (F, v)$.

* This assertion, as well as Proposition 1, was obtained independently of the author by A. Merzon.

** The closure of the set φU with respect to the topology $\sigma(F, F')$ may be larger than the closure of the same set in the topology v , even when U is an absolutely convex set.

Theorem 6. Let E and F be separated l.v.s., and let $\varphi : E \rightarrow F$ be a linear mapping with closed graph.

- a) If φ is almost continuous, and F is a perfectly complete l.v.s., then φ is continuous.
- b) If φ is almost open, $\varphi E = F$, and E is a perfectly complete l.v.s., then φ is open.

Theorem 6 is proved with the aid of Theorem 5 according to the scheme proposed in ⁴, Ch. VI, § 2.

IV. Let E be a separated l.v.s. We shall say ⁶ that a set $A \subset E$ satisfies the double-limit condition if, for every neighborhood of zero U in E , and for any two sequences $\{e_i\}_{i=1}^{\infty} \subset A$ and $\{f_j\}_{j=1}^{\infty} \subset U^0$, from the existence of the double limits $\lim_i \lim_j \langle e_i, f_j \rangle$ and $\lim_j \lim_i \langle e_i, f_j \rangle$ it follows that they are equal.

Proposition 3. Let E be a complete separated l.v.s., and let $A \subset E$ satisfy the double-limit condition. Then A is $\sigma(E, E')$ -precomplete.

Theorem 7 (Eberlein). Let E be a separated complete l.v.s., and let A be a subset such that every sequence of it has a weak cluster point. Then $\overline{A}^{\sigma(E, E')}$ is weakly compact.

With the aid of Theorem 5 it is proved that A satisfies the double-limit condition.

Proposition 4. Let B be a $\sigma(E, E')$ -precomplete absolutely convex subset of a separated l.v.s. E . Then B satisfies the double-limit condition.

This assertion is also proved by applying Theorem 5.

From Propositions 3 and 4 it follows:

Theorem 8. An absolutely convex subset A of a complete separated l.v.s. E is $\sigma(E, E')$ -precomplete if and only if it satisfies the double-limit condition.

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