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1969

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Abstract

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UDC 519.24/27

MATHEMATICS

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APPROXIMATELY OPTIMAL PROPERTIES OF WALD TESTS AND THE PROBLEM OF TESTING STATISTICAL HYPOTHESES

(Presented by Academician Yu. V. Linnik on 7 IV 1969)

The content of the note is adjacent to the work of A. Wald⁽¹⁾, where a number of approximately optimal properties of tests based on the maximum likelihood estimate were proved; in our work we shall call these Wald tests. The accuracy in Wald's theorems is estimated; it is shown that it has order $O(N^{-1/2+\varepsilon})$, where N is the sample size and ε is any small positive number.

I. Let X_n ($n = 1, \dots, N$) be independent identically distributed P -measure random vectors having density $f(X, \theta)$, which satisfies the following conditions:

A. θ belongs to a nondegenerate closed interval Ω of the Euclidean space E_K , and third-order partial derivatives with respect to θ_i ($i = 1, \dots, K$), $\theta^T = (\theta_1, \dots, \theta_K)$, exist almost everywhere on the space X for all θ . (Here and below the indices i, j, l run through the set of numbers $1, \dots, K$.)

B. $|\partial f(X, \theta)/\partial \theta_i| < F_1(X)$, $|\partial^2 f(X, \theta)/\partial \theta_i \partial \theta_j| < F_2(X)$, where F_1 and F_2 are integrable on $(-\infty, +\infty)$.

C. Almost everywhere on the space X there exist $\partial^2 \ln f(X, \theta)/\partial \theta_i \partial \theta_j$ and $\partial^3 \ln f(X, \theta)/\partial \theta_i \partial \theta_j \partial \theta_l$, and $\| -E_\theta \partial^2 \ln f(X_n, \theta)/\partial \theta_i \partial \theta_j \|$ is a positive definite matrix with absolute values of the determinant (for all $\theta \in \Omega$) not less than K_2 (here and below K_i, m_i , $i = 0, 1, 2, \dots$, denote fixed positive numbers).

D. For fixed $\theta^0 \in \Omega$ and for any $\theta \in \Omega$

$$|\theta - \theta^0| < \frac{\ln^{m_0} N}{\sqrt{N}} \left(|\theta - \theta^0|^2 = \sum_{i=1}^K |\theta_i - \theta_i^0|^2 \right) :$$

a) There exists a number a ($0 < a < 1/6$) such that

$$E_\theta \exp \left| \frac{\partial \ln f(X_n, \theta)}{\partial \theta_i} \right|^{4a/(2a+1)} < K_3.$$

b) There exists a number N_0 such that for $N = N_0$ the random vector

$$\left(\frac{1}{\sqrt{N}} \sum_{n=1}^N \frac{\partial \ln f(X_n, \theta)}{\partial \theta_i} \right)$$

has density $p_{0N_0}(y)$ and $\sup_{y \in E_K} |p_{0N_0}(y)| < K_4$.

$$\text{E. } E_{\theta_1} \left| \frac{\partial^3 \ln f(X_n, \theta^2)}{\partial \theta_i \partial \theta_j \partial \theta_l} \right|^2 \text{ and}$$

$$\int_{-\infty}^{+\infty} \dots \int \left| \frac{\partial^2 \ln f(X, \theta^1)}{\partial \theta_i \partial \theta_j} \frac{\partial f(X, \theta^2)}{\partial \theta_l} \right| dX_1 \dots dX_p$$

are bounded for $\theta^1, \theta^2 \in \Omega$ and $|\theta^1 - \theta^2| < \ln^{m_2} N / \sqrt{N}$.

II. It is proved that there exists a maximum likelihood estimate $\hat{\theta}^N(X)$ for θ and, as $N \rightarrow \infty$,

$$P_{\theta} \{ \sqrt{N} |\hat{\theta}^N(x) - \theta| > \ln^4 N \} = O(N^{-\sqrt{\ln N}}). \quad (1)$$

Consider the system of equations

$$\frac{1}{\sqrt{N}} \sum_{n=1}^N \frac{\partial \ln f(X_n, \theta)}{\partial \theta_i} + \sum_{j=1}^K \sqrt{N} (\dot{\theta}_j^N(X, \theta) - \theta_j) E_{\theta} \frac{\partial^2 \ln f(X_n, \theta)}{\partial \theta_i \partial \theta_j} = 0 \quad (2)$$

$$(i = 1, \dots, K);$$

in C the system has a solution $\dot{\theta}^N(X, \theta)$, which we shall call an auxiliary function for the maximum-likelihood estimate. It can be verified that

$$\sup_{z_0 \in E_K} |p_{\theta N}(z_0) - q_{\theta}(z_0)| = O(N^{-1/2+\varepsilon}), \quad (3)$$

where $p_{\theta N}(z_0)$ is the density ($z_0 = \sqrt{N} \dot{\theta}^N(X, \theta^0)$) under the true θ ; $q_{\theta}(z_0)$ is the density of the normal law $N(\sqrt{N}\theta, \|C_{ij}(\theta^0)\|^{-1})$;

$$|\theta - \theta^0| < \ln^{m_0} N / \sqrt{N}, \quad C_{ij}(\theta^0) = -E_{\theta} \partial^2 \ln f(X_n, \theta^0) / \partial \theta_i \partial \theta_j.$$

For the subsequent arguments we shall need the equality

$$P_{\theta} \left\{ \left| \frac{1}{N} \sum_{n=1}^N \frac{\partial^2 \ln f(X_n, \theta)}{\partial \theta_i \partial \theta_j} - E_{\theta} \frac{\partial^2 \ln f(X_n, \theta)}{\partial \theta_i \partial \theta_j} \right| > \frac{\ln^5 N}{\sqrt{N}} \right\} = O(N^{-1} \sqrt{\ln N}). \quad (4)$$

Assumption G. For a given $\varepsilon > 0$ there exists a number N_{ε} such that, for $N > N_{\varepsilon}$, relations (1), (3), (4) hold for all $\theta, \theta^0 \in \Omega$ with

$$|\theta - \theta^0| < \ln^m N / \sqrt{N},$$

where m is a fixed positive number.

III. We first consider a linear hypothesis. Here θ is written in the form

$$\theta = (\theta_1, \dots, \theta_K)^T = ({}_1\theta, {}_2\theta)^T,$$

where

$${}_1\theta^T = (\theta_1, \dots, \theta_r), \quad r < K.$$

As in (1), the problem of testing the linear hypothesis is posed in the form

$$H_1 : {}_1\theta = {}_1\theta^0,$$

where ${}_1\theta^0$ is known ($\omega = \{\theta : {}_1\theta = {}_1\theta^0\}$), against the alternative

$$K_1 : {}_1\theta \neq {}_1\theta^0.$$

Let $\underline{\theta}$ be a fixed point belonging to ω ; C a constant > 0 , and $S_C(\underline{\theta}, \theta)$ the surface

$$({}_1\theta - {}_1\theta^0)^T \|\bar{C}_{ij}(\theta)\| ({}_1\theta - {}_1\theta^0) = C, \quad \gamma(\theta)(\theta - \underline{\theta}) = 0, \quad (5)$$

where

$$\|\bar{C}_{ij}(\theta)\| = \|\sigma_{ij}(\theta)\|^{-1} \quad (i, j = 1, \dots, r),$$

$$\|\sigma_{ij}(\theta)\| = \|C_{ij}(\theta)\|^{-1} \quad (i, j = 1, \dots, K);$$

$\|C_{ij}(\theta)\|$ is the information matrix defined in C , and $\beta(\theta)$ and $\gamma(\theta)$ are $r \times r$ and $(K - r) \times K$ matrices, respectively, such that

$$\left\| \begin{pmatrix} \beta(\theta) & 0 \\ \gamma(\theta) & \end{pmatrix} \right\|^T \cdot \|C_{ij}(\theta)\| \cdot \left\| \begin{pmatrix} \beta(\theta) & 0 \\ \gamma(\theta) & \end{pmatrix} \right\| = E;$$

E is the identity matrix. Transform $S_C(\theta, \underline{\theta})$ into the sphere

$$({}_1\theta' - {}_1\theta^0)^T({}_1\theta' - {}_1\theta^0) = C, \quad {}_2\theta' = {}_2\theta^0.$$

Let

$$\eta(\theta) = \lim_{\rho \rightarrow 0} A(\omega'(\theta, \rho)) / A(\omega(\theta, \rho)), \quad (6)$$

where

$$\omega(\theta, \rho) = \{\bar{\theta} : \bar{\theta} \in S_C(\underline{\theta}, \theta), |\bar{\theta} - \theta| < \rho\},$$

ω' is the image of ω ;

$$A(\omega) = \int_{\omega} dA.$$

Theorem 1. The Wald test W'_N

$$N(\hat{\theta}^N(X) - {}^1\theta^0)^T \|\bar{C}_{ij}(\hat{\theta}^N(X))\| ({}^1\hat{\theta}^N(X) - {}^1\theta^0) > d_N \quad (7)$$

with level α (among all nonrandomized tests) as $N \rightarrow \infty$:

- 1) has approximately best average power on the family of surfaces $\{S_C(\theta, \underline{\theta})\}$ (5) with weight $\eta(\theta)$ (6), with accuracy up to $O(N^{-1/2+\varepsilon})$,

$$\sup_{S_C(\theta, \underline{\theta}), Z_N} \left\{ \int_{S_C(\theta, \underline{\theta})} E_{\theta} Z_N \frac{\eta(\theta)}{AS_C(\theta, \underline{\theta})} dA - \int_{S_C(\theta, \underline{\theta})} E_{\theta} W'_N \frac{\eta(\theta)}{AS_C(\theta, \underline{\theta})} dA \right\} = O(N^{-1/2+\varepsilon}), \quad (8)$$

where Z_N is any test of level α , $AS_C = \int_{S_C} \eta(\theta) dA$.

- 2) has approximately best constant power on the family $\{S_C(\theta, \underline{\theta})\}$ with accuracy up to $O(N^{-1/2+\varepsilon})$;
- 3) is an approximately most stringent test with accuracy up to $O(N^{-1/2+\varepsilon})$;
- 4) is an approximately minimax test with accuracy up to $O(N^{-1/2+\varepsilon})$.

The theorem is proved with the aid of an auxiliary hypothesis on the space $\hat{\theta}^N(X, \theta^0)$.

If the density of the distribution belongs to the normal law $N(\xi, \Sigma)$, rewrite it in the form $f(X, \theta)$, where θ is a parameter and $\theta^T = (\xi_1, \dots, \xi_p, \sigma_{11}, \dots, \sigma_{pp})$; we again obtain the assertion of Theorem 1—among all randomized tests. As a consequence, the approximately optimal properties of the test T^2 hold, and by the method of proof of the theorem formulated above the approximately

optimal properties also hold for the test R^2 (under certain restrictions on Ω); these results agree with (5, 7).

IV. We shall consider a general composite hypothesis, where the region ω is given by the equations

$$\xi_1(\theta) = \dots = \xi_r(\theta) = 0, \quad r < K. \quad (9)$$

Under certain conditions there exist $K - r$ functions $\xi_{r+1}(\theta), \dots, \xi_K(\theta)$, and θ is a function of $(\xi_1, \dots, \xi_K)$.

Let

$$f^*(X, \xi) = f(X, \theta(\xi))$$

satisfy assumptions A, B, C, D, E, F with respect to ξ .

Define

$$C_{ij}^*(\theta) = -E_{\xi} \partial^2 \ln f^*(X_n, \xi) / \partial \xi_i \partial \xi_j. \quad (10)$$

Let $\theta \in \omega$ and let C be a fixed number; construct the surface $S_C^*(\theta, \underline{\theta})$ and define the weight $\eta^*(\theta)$ with the help of $C_{ij}^*(\theta)$ as in (5).

Theorem 2. For testing the hypothesis $H_2 : \theta \in \omega$ against the alternative $K_2 : \theta \in \Omega - \omega$ with fixed level α . Among all nonrandomized tests the Wald test W_N^2

$$N({}^1\xi(\hat{\theta}^N(X)))^T \|\bar{C}_{ij}^*(\hat{\theta}^N(X))\|({}^1\xi(\hat{\theta}^N(X))) > d_N, \quad (11)$$

where $({}^1\xi(\theta))^T = (\xi_1(\theta), \dots, \xi_r(\theta))$, d_N is determined by the level α :

- 1) has approximately best average power on the family of surfaces $\{S_C^*(\theta, \underline{\theta})\}$ with weight $\eta^*(\theta)$, with accuracy up to $O(N^{-1/2+\varepsilon})$;
- 2) has approximately best constant power on the family $\{S_C^*(\theta, \theta)\}$ with accuracy $O(N^{-1/2+\varepsilon})$;
- 3) is an approximately most stringent test with accuracy up to $O(N^{-1/2+\varepsilon})$;
- 4) is an approximately minimax test with accuracy up to $O(N^{-1/2+\varepsilon})$.

Theorem 2 remains valid when the Wald test in (11) is replaced by the likelihood-ratio test under certain restrictions.

In conclusion, I express my deep gratitude to Academician Yu. V. Linnik for posing the problem and for his attention to the work; I also thank N. M. Mitrofanova, A. M. Kagan, and V. V. Petrov for valuable consultations.

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Received
7 III 1969

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Note: Figure translations are in progress. See original paper for figures.

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