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Abstract

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MATHEMATICS

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ON THE SUPERPOSITION OF ABSOLUTELY CONTINUOUS FUNCTIONS AND ON THE SUPERPOSITION OF FUNCTIONS OF BOUNDED VARIATION

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The question of the superposition of absolutely continuous functions was studied in detail in the work of G. M. Fichtenholz ⁽¹⁾ and in the works of N. K. Bari (see the detailed exposition ⁽²⁾), who also investigated the question of the superposition of functions of bounded variation ⁽³⁾.

In ⁽¹⁾ it was established, in particular, that in order that, for every absolutely continuous function $\Phi(x)$, the superposition $\Phi(f(x))$ be absolutely continuous, it is necessary and sufficient that the function $f(x)$ be absolutely continuous and that the solution of the equation $f(x) = y_0$ consist of isolated points and intervals, the total number of which does not exceed a number N independent of y_0 . It follows from this that, by improving the differential properties of the function $f(x)$, we cannot, generally speaking, ensure that the superposition $\Phi(f(x))$ becomes absolutely continuous. In particular, one can choose Φ and f so that f is infinitely differentiable, Φ is an absolutely continuous function, but the superposition $\Phi(f(x))$ is not absolutely continuous. An analogous example can also be constructed in the case when Φ is a function of bounded variation.

On the other hand, it is known that the product $\Phi(f(x))f'(x)$ may possess better properties than the factor $\Phi(f(x))$. For example, if $\Phi(x)$ is a summable function and $f(x)$ is absolutely continuous and monotone, then the product $\Phi(f(x))f'(x)$ is a summable function, although $\Phi(f(x))$ may fail to be a summable function (see, for example, ⁽⁴⁾, p. 284). Let us also note that if $\Phi(x)$ is a bounded measurable function and $f(x)$ is absolutely continuous, then $\Phi(f(x))f'(x)$ is a summable function (⁽⁵⁾, p. 302).

In the present paper it is clarified to what extent the properties of the product $\Phi(f(x))f'(x)$ improve in comparison with the factor $\Phi(f(x))$ in the case when Φ is an absolutely continuous function or a function of bounded variation.

In what follows we shall assume that the function $f(x)$ is defined on the interval

$[a, b]$, and $\Phi(x)$ on the interval $[c, d]$, where

$$c = \min_{x \in [a, b]} f(x), \quad d = \max_{x \in [a, b]} f(x).$$

Theorem 1. Let $\Phi(x)$ be absolutely continuous on $[c, d]$, and let $f(x)$ be differentiable on $[a, b]$ and let the derivative $f'(x)$ be absolutely continuous. Then the function $\Phi(f(x))f'(x)$ is absolutely continuous on $[a, b]$.

It has already been noted above that the factor $\Phi(f(x))$ need not be absolutely continuous. We illustrate this fact by a simple example. Let

$$f_{\alpha, \beta}(x) = x^\alpha \left(\sin \frac{1}{x^\beta} + 2 \right)$$

for $0 < x \leq 1$, $f_{\alpha, \beta}(0) = 0$, where $\alpha > 0$, $\beta > 0$, and $\Phi(x) = x^\gamma$, $0 \leq x \leq 3$, $0 < \gamma < 1/2$. It is not hard to verify that if one chooses $\beta > \gamma/(1 - 2\gamma)$ and $1 + 2\beta < \alpha \leq \beta/\gamma$, then the derivative $f'(x)$ is absolutely continuous; consequently, by Theorem 1 the product $\Phi(f(x))f'(x)$ is absolutely continuous, whereas the factor $\Phi(f(x))$ is not an absolutely continuous function.

Below we give a less trivial example, which shows to what extent “bad” differential properties may be possessed by the factor $\Phi(f(x))$ in Theorem 1. Let us note that, by virtue of Banach’s theorem ⁽⁶⁾, p. 169, the factor $\Phi(f(x))$ in Theorem 1 has a derivative on a set $E \subset [a, b]$ such that $\text{mes } E \cap (\alpha, \beta) > 0$ for every interval $(\alpha, \beta) \subset [a, b]$. On the other hand, for any perfect nowhere dense set $P \subset [a, b]$ (whose measure may be arbitrarily close to $b - a$) one can specify Φ and f , satisfying the condition of Theorem 1, such that the superposition $\Phi(f(x))$ is not differentiable on the set P (in particular, if $\text{mes } P > 0$, then the function $\Phi(f(x))$ is certainly not absolutely continuous).

The functions Φ and f are constructed as follows. Let $\varphi(x) \in C^\infty[0, 1]$, $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$, $n = 0, 1, \dots$, $0 < \varphi(x) \leq 1$ for $x \in (0, 1)$, $\varphi(1/2) = 1$.

Let, further,

$$(a, b) - P = \sum_{k=1}^{\infty} (a_k, b_k), \quad \delta_k = b_k - a_k, \quad \text{where } \delta_1 \geq \delta_2 \geq \dots$$

Denote by r_n the length of the largest of the intervals of the set

$$[a, b] - \sum_{k=1}^n (a_k, b_k).$$

It is not difficult to prove that $r_n \downarrow 0$ as $n \rightarrow \infty$. Put $f(x) = 0$ on P and $f(x) = \delta_k^2 \varphi[(x - a_k)/\delta_k]$ on (a_k, b_k) . As $\Phi(x)$, $0 \leq x \leq 1$, take a monotonically increasing function, differentiable for all $0 < x \leq 1$, for which $\Phi(0) = 0$, $\Phi(\delta_n^2) = \delta_n + r_n$. In the case when f is an absolutely continuous function, an example of analogous content was constructed by N. K. Bari ⁽²⁾, p. 212.

A theorem analogous to Theorem 1 is also valid in the case when Φ is a function of bounded variation.

Theorem 2. Let $\Phi(x)$ have bounded variation on $[c, d]$, $f(x)$ be differentiable on $[a, b]$, and the derivative $f'(x)$ have bounded variation. Then the function $\Phi(f(x))f'(x)$ has bounded variation.

It should be noted that if $\Phi(x)$ is a summable function, then the theorem analogous to Theorems 1 and 2 is false for it (although, as was noted at the beginning of the article, an improvement of the properties of the product $\Phi(f(x))f'(x)$ in comparison with the factor $\Phi(f(x))$ is observed). This can be verified by again considering the functions $f(x) = f_{\alpha, \beta}(x)$ and $\Phi(x) = x^\gamma$. If $-1 < \gamma < 0$ and $1 < \beta < \alpha \leq \beta/(1 + \gamma)$, then $\Phi(x)$ is summable, the derivative $f'(x)$ exists everywhere on $[0, 1]$ and is summable, but the product $\Phi(f(x))f'(x)$ is not summable.

The proof of Theorem 2 consists of two parts. First it is proved that

$$\text{Var}_{[a, b]} \Phi(f(x))f'(x) \leq M \text{Var}_{[a, b]} f'(x) + \sum_i \text{Var}_{[a_i, b_i]} \Phi(f(x))f'(x), \quad (1)$$

where

$$M = \sup_{x \in [a, b]} |\Phi(f(x))|,$$

and the intervals $[a_i, b_i]$ are the closures of the component intervals (a_i, b_i) of the open set

$$S = \{x : x \in (a, b), f'(x) \neq 0\}.$$

Next, the second term on the right-hand side of inequality (1) is estimated. As a result we arrive at the estimate

$$\begin{aligned} \text{Var}_{[a, b]} \Phi(f(x))f'(x) \leq & \begin{cases} M \text{Var}_{[a, b]} f'(x) + m \text{Var}_{[c, d]} \Phi(x), & \text{if } f'(x) \neq 0 \text{ on } [a, b], \\ [2M + \text{Var}_{[c, d]} \Phi(x)] \text{Var}_{[a, b]} f'(x), & \text{otherwise} \end{cases} \\ & \left(m = \sup_{x \in [a, b]} |f'(x)| \right), \end{aligned} \quad (2)$$

from which Theorem 2 follows.

For the proof of Theorem 1 we use the Banach-Zarecki theorem (see ⁽⁷⁾ or ⁽⁴⁾, p. 269), according to which a continuous function $\varphi(x)$, $x \in [a, b]$, of bounded variation is absolutely continuous if and only if

and only if it has property N on $[a, b]$, i.e., for any $M \subset [a, b]$, $\text{mes } M = 0$ implies $\text{mes } \varphi(M) = 0$. Theorem 1 follows from this theorem, Theorem 2, and the following assertion: if f and g are continuous and have property N , then the product fg also has property N .

Theorem 1 can be used in certain questions on equations in partial derivatives in the following situation (we give the simplest example). In a number of cases it is necessary to integrate by parts the expression

$$\int_a^b \frac{d^2 u}{dx^2} v dx,$$

where the function $u(x)$ is twice continuously differentiable, and $v(x)$ is chosen in a special way. Sometimes it is convenient to put $v(x) = \Phi(u(x))$, where Φ is an absolutely continuous function (for example, $\Phi(t) = (|t|^p)'$, $1 < p < \infty$; here the case $1 < p < 2$ is of interest). The superposition $\Phi(u(x))$ need not be absolutely continuous. In this case we shall use the fact that the formula for integration by parts

$$\int_a^b f' g dx = fg|_a^b - \int_a^b fg' dx$$

is valid under the following assumptions: 1) f' and g' exist almost everywhere, 2) one of the products $f'g$ or fg' is summable, and 3) the product fg is absolutely continuous. In the case under consideration $f = du/dx$, $g = \Phi(u)$. By Theorem 1 the product $fg = \Phi(u) du/dx$ is absolutely continuous. Therefore

$$\int_a^b \frac{d^2 u}{dx^2} \Phi(u) dx = \Phi(u) \frac{du}{dx} \Big|_a^b - \int_a^b \Phi'(u) \left(\frac{du}{dx} \right)^2 dx$$

(if it is known that almost everywhere $(\Phi(u))' = \Phi'(u)u'$).

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