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Abstract

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MATHEMATICS

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE REGULARITY OF A BOUNDARY POINT FOR THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

(Presented by Academician I. G. Petrovskii, 19 VII 1968)

In this note necessary and sufficient conditions will be obtained for the regularity of a boundary point for the Dirichlet problem for the equation

$$Lu \equiv \sum_{i,k=1}^n a_{ik}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^n b_i(t, x) \frac{\partial u}{\partial x_i} + c(t, x)u - \frac{\partial u}{\partial t} = 0 \quad (1)$$

in an arbitrary domain D of the space t, x , $x = (x_1, \dots, x_n)$.

With respect to the coefficients of the equation it is assumed that:

$$C_1 |\xi|^2 \leq \sum_{i,k=1}^n a_{ik} \xi_i \xi_k \leq C_2 |\xi|^2, \quad C_1, C_2 > 0, \quad (2)$$

all the coefficients are bounded and satisfy a Hölder condition.

In the case when the domain D is a cylinder, for the heat equation this was done by A. N. Tikhonov ⁽¹⁾. For the one-dimensional heat equation and for a domain bounded by two straight lines parallel to the t -axis and by the curves $x = \varphi_1(t)$ and $x = \varphi_2(t)$, sufficient conditions and necessary conditions, very close to one another but not completely coinciding, were obtained by I. G. Petrovskii ⁽²⁾. In the multidimensional case, the question of the regularity of a boundary point when at this point the boundary is tangent to the characteristics was not considered for a long time. In recent years interest in these questions has revived. On the one hand, the behavior of the solution of the Dirichlet problem for a parabolic equation began to be studied at the very highest and the very lowest, with respect to the t -axis, points of the domain, when in neighborhoods of these points the boundary behaves like a paraboloid ^(3,4). On the other hand, the question of the regularity of a boundary point for a parabolic equation began to be considered as a special case of the more general problem of

degenerate elliptic-parabolic equations ^(4,6,7), etc. In these works the boundary was assumed sufficiently smooth and, in particular, the question of the relation between the smoothness of the solution at such a point and the character of the boundary was investigated. In the present note an arbitrary boundary is considered, but only the question of the continuity of the solution at a boundary point is studied.

1°. Some definitions and notation. A domain D^0 , whose boundary Γ^0 is a polyhedron, will be called **simple**.

For a simple domain consider the Dirichlet problem

$$Lu = 0 \text{ in } D^0, \quad u|_{\Gamma^0} = \varphi, \quad \varphi \text{ continuous on } \Gamma^0. \quad (3)$$

By a solution of such a problem we shall mean a function satisfying equation (1) in D^0 , continuous on Γ^0 except for the $(n+1-k)$ -dimensional faces ($k = 1, \dots, n+1$) orthogonal to the t -axis (if $k = n+1$, i.e. the face is a vertex, then we shall include here those vertices which have a supporting hyperplane orthogonal to the t -axis), which cannot be touched from inside the domain from above, bounded and assuming the boundary val-

φ everywhere, except on the indicated faces. Such a solution exists and is unique.

Let an arbitrary bounded domain D with boundary Γ be given, and consider for it the Dirichlet problem:

$$Lu = 0, \quad u|_{\Gamma} = \varphi, \quad \varphi \text{ is continuous on } \Gamma. \quad (4)$$

We construct a generalized solution of this problem by Wiener's method: approximate D from inside by simple domains D_m^0 . Extend φ continuously to $\bar{D}$. Consider the sequence of problems

$$Lu_m = 0, \quad u_m|_{\Gamma_m^0} = \varphi$$

(Γ_m^0 is the boundary of D_m^0). The sequence $\{u_m\}$ converges at each point of D to a certain function $u(t, x)$. The function $u(t, x)$ is a solution of equation (1), is bounded, and depends neither on the method of extending the function φ nor on the choice of the sequence $\{D_m^0\}$. This function is called the **generalized solution of the Dirichlet problem** (4).

A point (t, x) of the boundary Γ of the domain D is called **regular** for the Dirichlet problem for equation (1) if, for every continuous φ prescribed on the boundary, the generalized solution of problem (4) assumes at this point its boundary value.

Let E be a B -set in the $(n+1)$ -dimensional space (t, x) . Let the point (t^0, x^0) be such that E lies entirely in the half-space $t < t^0$. Consider all possible measures μ , defined on E , such that

$$\int_E G(t, x; \tau, \xi) d\mu(\tau, \xi) \leq 1 \quad \text{for } (t, x) \notin E, \quad (5)$$

where

$$G(t, x; \tau, \xi) = \begin{cases} \frac{1}{(t - \tau)^{n/2}} \exp \left[-\frac{|x - \xi|^2}{4(t - \tau)} \right], & \text{for } t > \tau, \\ 0, & \text{for } t \leq \tau. \end{cases}$$

We shall call the number

$$\gamma(E, t^0, x^0) = \sup \int_E G(t^0, x^0; \tau, \xi) d\mu(\tau, \xi)$$

the **potential of the set E at the point (t^0, x^0)** , where the least upper bound is taken over all measures satisfying condition (5).

Define two sequences of numbers $\{\rho_m\}$ and $\{\tau_m\}$. Define the sequence $\{\rho_m\}$ recursively:

$$\rho_0 = 1/2, \quad \rho_{m+1} = \rho_m / \sqrt{|\ln \rho_m|}.$$

Define the numbers τ_m through ρ_m from the condition

$$4n\tau_m |\ln \tau_m| = \rho_{m-2}^2.$$

Let the point (t^0, x^0) belong to the boundary of the domain D . Denote by H the complement of the domain D , and let H_m be the intersection of H with the strip

$$t^0 - \tau_m \leq t < t^0 - \tau_{m+1}.$$

Denote by γ_m the potential of the set H_m with respect to the point (t^0, x^0) .

2°. Theorem. *In order that the point $(t^0, x^0) \in \Gamma$ be a regular boundary point for problem (4), it is necessary and sufficient that*

$$\sum_{m=1}^{\infty} \gamma_m = \infty. \quad (6)$$

We shall give here the main points of the proof for the case when (1) is the equation

$$\Delta u - \partial u / \partial t = 0. \quad (7)$$

For convenience we shall assume that (t^0, x^0) coincides with the origin of coordinates.

Introduce the notation: $n' = \max(n, 3)$; Π_m is the cylinder $|x| < \rho_{m-n}$, $-\tau_m \leq t < 0$; Π_m^1 is the cylinder $|x| < \rho_m$, $-\tau_m \leq t < -\tau_{m+1}$; Π_m^2 is the cylinder $|x| < \rho_{m+n}$, $-\tau_{m+n} \leq t < 0$; S_m^1 is the lateral surface of the cylinder Π_m^1 .

Lemma 1. Put $E_m = H_m \setminus \Pi_m^1$. Then

$$\sum_{m=1}^{\infty} \gamma(E_m, 0, 0) < \infty. \quad (8)$$

The proof of this lemma is obtained by a simple calculation.

Let D^0 be a domain contained in D , intersecting the lateral surface or the lower base of the cylinder Π_m . Let $D_m^0 = D^0 \cap \Pi_m$, $D_m^2 = D^0 \cap \Pi_m^2$, $H_m^1 = H_m \cap \Pi_m^1$. Let μ be a measure defined on H_m^1 and such that

$$U_m(t, x) = \int_{H_m^1} G(t, x; \tau, \xi) d\mu(\tau, \xi) \leq 1 \quad \text{for } (t, x) \in H_m^1. \quad (9)$$

Let $U_m(0, 0) = \gamma_m^0$.

Lemma 2. Let in D_m there be defined a solution u of equation (7), positive in D_m^0 , continuous in D_m^0 and vanishing on that part Γ_m^0 of the boundary D_m^0 which is situated strictly inside Π_m . Then

$$\sup_{(t,x) \in D_m^0} u(t, x) > (1 + \eta \gamma_m^0) \sup_{(t,x) \in D_m^2} u(t, x) \quad (10)$$

for $m > m_0$, where $\eta > 0$ and m_0 depend on n .

We give the main points of the proof of this lemma. It is shown that, for sufficiently large m ,

$$\sup_{(t,x) \in S_m^1} U_m(t, x) \leq \frac{1}{2} \inf_{(t,x) \in \Pi_m^2} U_m(t, x), \quad (11)$$

$$\inf_{(t,x) \in \Pi_m^2} U_m(t, x) \geq \frac{1}{2} \gamma_m^0. \quad (12)$$

Put $\sup_{(t,x) \in \Pi_m} u(t, x) = M$ and construct the auxiliary function

$$v = M \left[1 - U_m + \frac{1}{2} \inf_{\Pi_m^2} U_m \right].$$

We then have $v|_{S_m} \geq u|_{S_m}$ (by (9) and (11)); $v|_{\Gamma_m^0} \geq u|_{\Gamma_m^0}$ (since $v|_{\Gamma_m^0} > 0$ and $u|_{\Gamma_m^0} = 0$); and, finally, for $t = -\tau_m$ outside H_m^1 we also have $u > v$, since there $U_m = 0$ and therefore $v > M$. Hence, by the maximum principle, $v > u$ in D_m^0 , and therefore

$$\sup_{\Pi_m^2} u \leq M \left(1 - \frac{1}{2} \inf_{\Pi_m^2} U_m \right),$$

which, taking into account (11) and (12), gives the required inequality (9).

To prove the regularity of the point $(0, 0)$, it is enough to show the following: let ε and K be two positive numbers and let m' be a natural number. There exists $\delta > 0$ such that, whatever the simple domain D^0 intersecting the δ -neighborhood of the origin and the hyperplane $t = -\tau_{m'}$, and whatever the solution $u(t, x)$ of equation (7), bounded above by the constant K and vanishing on that part of the boundary of D^0 which is situated in the open strip $-\tau_{m'} < t < 0$, inside the intersection of D^0 with the δ -neighborhood of the origin, the inequality will hold—

inequality $u(t, x) < \varepsilon$. From the divergence of the series (6) and Lemma 1 it follows that at least one of the series $\sum_{m=1}^{\infty} \gamma_m^0 + (2n' + l)m$, $l = 0, 1, \dots, 2n-1$, diverges. Let, for definiteness, this be the series $\sum_{m=1}^{\infty} \gamma_{m'+2mn'}^0$. Let m'' be so large that

$$\prod_{m=1}^{m''} (1 + \eta \gamma_{m'+2n'm}^0) > \frac{K}{\varepsilon}. \quad (13)$$

If now we assume that the intersection D^0 with $\Pi_{m'+2n'm''}$ is nonempty and that

$$\sup_{(t,x) \in D_{m'+2n'm''}^0} u(t, x) > \varepsilon, \quad (14)$$

then we arrive at a contradiction: successively applying Lemma 2 to pairs of cylinders $\Pi_{m'+2n'm}$ and $\Pi_{m'+2n'(m-1)}$, we obtain

$$\sup_{\Pi_{m'+2n'(m-1)}} u > (1 + \eta \gamma_{m'+2n'm}^0) \sup_{\Pi_{m'+2n'm}} u,$$

and therefore from (13) and (14) it follows that $\sup u > K$.

3°. We give the scheme of the proof of the necessity of condition (6).

Choose m_1 so large that

$$\sum_{m_1}^{\infty} \gamma_m < \frac{1}{4}.$$

Consider the intersection D_m of the domain D with the strip $-\tau_m < t < -\tau_{m+1}$. It can be shown that in D_m one can inscribe a simple domain D'_m such that, if by H'_m we denote the complement of D'_m in this strip, then the difference $\gamma_m - \gamma(H'_m, 0, 0)$ will be arbitrarily small. Consider those n -dimensional faces of the boundary of D'_m which are situated on the hyperplane $t = -\tau_{m+1}$. Denote by D''_m that part of D'_m which consists of points each of which can be connected with one of the indicated faces by means of a polygonal line lying in D'_m and projecting one-to-one onto the t -axis. Let Γ''_m be the boundary of D''_m . Let K''_m be the aggregate of the n -dimensional faces of Γ''_m lying in the hyperplane $t = -\tau_m$. Remove from Γ''_m these faces, and also the faces orthogonal to the t -axis which cannot be touched from D''_m from above, and denote the remaining part of Γ''_m by S''_m . In D''_m consider the solution v_m of the following first boundary-value problem

$$v_m|_{S''_m} = 1, \quad v_m|_{K''_m} = 0.$$

The function v_m can be represented by means of a simple-layer potential with density concentrated on S''_m . Moreover $v_m(0, 0) \leq \gamma(H'_m, 0, 0)$, and we can obtain

$$\sum_{m=m_1}^{\infty} v_m(0, 0) < \frac{1}{2}.$$

After this, with the aid of the maximum principle it is shown that if the boundary function φ of the original problem satisfies the condition $\varphi(t, x) = 0$ for $t < -\tau_{m_1}$, $\varphi(t, x) \leq 1$ for $-\tau_{m_1} \leq t \leq 0$, then

$$\lim_{(t,x) \rightarrow (0,0)} u(t, x) < \frac{1}{2}.$$

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Note: Figure translations are in progress. See original paper for figures.

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