

INFINITE SPECTRA OF TOPOLOGICAL SPACES AND THEIR LIMIT SPACES

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1969

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Abstract

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UDC 519.831

MATHEMATICS

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INFINITE SPECTRA OF TOPOLOGICAL SPACES AND THEIR LIMIT SPACES

(Presented by Academician P. S. Aleksandrov on 3 XII 1968)

The main task of this paper is to carry over the results obtained in (1) for finite spectra of topological spaces to the case of infinite spectra, i.e., to projection spectra* $S_X = \{N_\alpha, \overline{\alpha\alpha'}\}$, where α runs through a directed family** of all partitions of the space X , and N_α is the nerve of the covering α . By $\widehat{S}$ we denote the limit space of the spectrum S .

1. Basic concepts connected with coverings.

A covering α of a space X whose elements are $\varkappa$ -sets (i.e., canonical closed sets) with disjoint open kernels will be called a decomposition of the space X ; locally finite decompositions are called partitions (2). It is easily verified that partitions are point-finite decompositions possessing the property of conservativity (= the sum of any number of elements of the decomposition is closed).

Once and for all we agree to denote by $\mathfrak{z} = \{\alpha\}$ any directed family of decompositions of the given space X . A directed family of all, respectively of all finite, partitions of the space X will be denoted by ζ_X , respectively by $\varkappa_X^{***}$. The set of all $\varkappa$ -sets of the space X will be denoted by $\mathfrak{A}_X$.

If in X a family of decompositions $\mathfrak{z} = \{\alpha\}$ is given, then by $\mathfrak{A}_{\mathfrak{z}}$ we denote the set of all $\varkappa$ -sets that are elements of at least one decomposition $\alpha \in \mathfrak{z}$, so that

$$\mathfrak{A}_{\mathfrak{z}} = \bigcup_{\alpha \in \mathfrak{z}} \alpha.$$

If $\mathfrak{A}_{\mathfrak{z}} = \mathfrak{A}_X$, then the family $\mathfrak{z}$ is called **rich**. A rich family consisting of conservative decompositions is called a **rich conservative family** (such, for example, are the families ζ_X and $\varkappa_X$).

Definition 1. For a given family $\mathfrak{z} = \{\alpha\}$ of decompositions of a space X , we shall call a $\mathfrak{z}$ -system a centered system $\xi = \{A_\lambda\}$ consisting of sets $A_\lambda \in \mathfrak{A}_{\mathfrak{z}}$ and satisfying two conditions:

- (a) If $A_\lambda \in \xi \cap \alpha$ and $\alpha' > \alpha$, then in $\alpha' \cap \xi$ there is some $A_{\lambda'} \subseteq A_\lambda$.
- (b) If $A_\lambda \in \xi$, $A \supseteq A_\lambda$, $A \in \mathfrak{A}_{\mathfrak{z}}$, then $A \in \xi$.

If $\mathfrak{z} = \zeta_X$, respectively $\mathfrak{z} = \varkappa_X$, then condition (a) is equivalent to the following condition (a_ζ) , respectively $(a_\varkappa)$: if $A_\lambda \in \xi$ and $A_\lambda = \bigcup_\mu A_{\lambda\mu}$, where $\{A_{\lambda\mu}\}$ is a partition, respectively a finite partition, of the set A_λ , then some $A_{\lambda\mu} \in \xi$.

Definition 2. A space X is called complete with respect to a given family $\mathfrak{z} = \{\alpha\}$ of decompositions in it, or simply $\mathfrak{z}$ -complete, if every $\mathfrak{z}$ -system $\xi = \{A_\lambda\}$ has a nonempty intersection

$$\bigcap_{A_\lambda \in \xi} A_\lambda \neq \Lambda.$$

* The basic definitions and properties of spectra are assumed known (see (2, 1)).

** Any set of coverings $\{\alpha\}$ of a given space X is partially ordered: a covering α' follows the covering α if α' is inscribed in α ; only this natural order will be considered.

*** Unlike the families ζ_X and $\varkappa_X$, the set of all decompositions, as well as the set of all conservative decompositions, is generally not directed and therefore does not fit our notion of a family $\mathfrak{z}$.

Remark 1. If $\mathfrak{z} = \zeta_X$, respectively $\mathfrak{z} = \chi_X$, then instead of $\mathfrak{z}$ -system, $\mathfrak{z}$ -complete, etc., we say ζ -system, ζ -complete, respectively χ -system, χ -complete, etc. It can be shown that for semiregular spaces χ -completeness is equivalent to bicompleteness.

On the set $\Omega_{\mathfrak{z}}$ of all $\mathfrak{z}$ -systems we introduce a topology as follows. For $H \subseteq X$, denote by V_H the set of all $\mathfrak{z}$ -systems $\xi = \{A_\lambda\}$ satisfying the condition $A_\lambda \cap H \neq \Lambda$ for every $A_\lambda \in \xi$.

In the important special case when $\mathfrak{z}$ is a rich conservative family, the topology in $\Omega_{\mathfrak{z}}$ is given by an open base consisting of all V_H , where H runs through all canonical open sets in X . It is easy to verify that this is the so-called Wallman topology.

In the general case the topology in $\Omega_{\mathfrak{z}}$ is defined by means of neighborhoods: for arbitrary $\xi \in \Omega_{\mathfrak{z}}$ and arbitrary $\alpha \in \mathfrak{z}$, define the neighborhood $O_\alpha \xi$ as the set $V_{I\tilde{t}_\alpha}$, where $t_\alpha = \xi \cap \alpha$, $\tilde{t}_\alpha \subseteq X$ is the union of all $A_\lambda^\alpha \in t_\alpha$, and $I\tilde{t}_\alpha$ is the open kernel of the set $\tilde{t}_\alpha \subseteq X$.

The set $\Omega_{\mathfrak{z}}$, considered with this topology, is a T_0 -space containing the T_1 -subspaces $\omega_{\mathfrak{z}}$ and $\dot{\omega}_{\mathfrak{z}}$, consisting respectively of all maximal and all minimal $\mathfrak{z}$ -systems. We shall call a $\mathfrak{z}$ -system ξ **finitary** if $\xi \cap \alpha$ is finite for every $\alpha \in \mathfrak{z}$.

In $\Omega_{\mathfrak{z}}$ there is contained the subspace $\Omega_{\mathfrak{z}}^\varphi$ of all finitary $\mathfrak{z}$ -systems. In $\Omega_{\mathfrak{z}}^\varphi$, in turn, lie the T_1 -spaces $\omega_{\mathfrak{z}}^\varphi$ and $\dot{\omega}_{\mathfrak{z}}^\varphi$, consisting of maximal, respectively minimal finitary $\mathfrak{z}$ -systems.

The spaces $\Omega_{\mathfrak{z}}$, $\omega_{\mathfrak{z}}$, $\dot{\omega}_{\mathfrak{z}}$, $\Omega_{\mathfrak{z}}^\varphi$, $\omega_{\mathfrak{z}}^\varphi$, $\dot{\omega}_{\mathfrak{z}}^\varphi$ for $\mathfrak{z} = \zeta_X$ are denoted respectively by $\Omega_\zeta X$, $\omega_\zeta X$, $\dot{\omega}_\zeta X$; $\Omega_\zeta^\varphi X$, $\omega_\zeta^\varphi X$, $\dot{\omega}_\zeta^\varphi X$. For $\mathfrak{z} = \chi_X$ we have only three spaces: $\Omega_\chi X$, $\omega_\chi X$,

$\omega_X X$; here $\omega_X X$ is precisely the Wallman-Ponomarev space that we denoted by $\omega_X X$ in (1).

2. First group of basic results. We shall call a spectral parasite lying in the spectrum $S = \{N_\alpha, \omega_\alpha^{\alpha'}\}$ a projection spectrum $\Theta = \{\Theta_\alpha, \omega_\alpha^{\alpha'}\}$, $\Theta_\alpha \subseteq N_\alpha$, in which, for all α following some α_0 , the complexes Θ_α are infinite and all of them are elementary, i.e. every nonempty finite subset of vertices of any complex Θ_α is the skeleton of some simplex of this complex.

Definition 3. The spectrum S_X is called **correct** if no spectral parasite lies in it.

Theorem 1. If the spectrum S_X of a space X is correct, then $\tilde{S}_X = \omega_\zeta X$.

Theorem 2. For every ζ -complete semiregular ($= T_\zeta$ -) space X , the spectrum S_X is correct and $\tilde{S}_X = X$. Conversely, if the spectrum S_X of a space X is correct and $\tilde{S}_X = X$, then X is a ζ -complete T_ζ -space.

Corollary 1. For ζ -complete T_ζ -spaces X one has the equality $\omega_\zeta X = X$.

A semiregular (i.e. T_ζ) space X is called a $h\zeta$ -, respectively $h\lambda$ -space,* if the spectrum $\tilde{S}_X$ is correct. By Ponomarev's results, every paracompactum is an $h\lambda$ - (all the more an $h\zeta$ -) space.

$h\zeta$ -spaces are characterized (among all T_ζ -spaces) by the fact that, whatever the ζ -system ξ and partition α may be, the set $\xi \cap \alpha$ is finite.

Theorem 3. For any $h\lambda$ -space X , the space $\tilde{S}_X = \omega_\zeta X$ is an extension of the space X that is a ζ -complete $h\zeta$ -space. Moreover, the spectra S_X and $S_{\omega_\zeta X}$ are isomorphic; consequently, $\omega_\zeta \omega_\zeta X = \omega_\zeta X$.

These results contain the desired analogues of Theorems 1-4 of the paper (1).

3. Generalization of the notion of nerve and projection spectrum. Let a nonempty set E of elements be given, conventionally called

* A semiregular (i.e. T_ζ) space X is called (1) a T_λ -space if, for every neighborhood O_x of any point $x \in X$, there is a set P that is the intersection of a finite number of $\chi\alpha$ -sets such that $x \in P \subseteq O_x$.

vertices. By a generalized complex K we shall mean any subset of the set of all "simplexes," i.e., of all (not necessarily finite) subsets* of the set E , satisfying the condition of completeness: from $T \in K$, $T' \subseteq T$ it follows that $T' \in K$. Simplicial mappings of generalized complexes, as well as generalized projection spectra $S = \{K_\alpha, \omega_\alpha^{\alpha'}\}$ and all concepts pertaining to them, in particular the concept of a thread, are now defined automatically.

On the set $\hat{S}$ of all threads of the given generalized spectrum $S = \{K_\alpha, \omega_\alpha^{\alpha'}\}$ the usual topology is introduced: a neighborhood $O_{\alpha_0} \xi$ of the thread $\xi = \{t_\alpha\}$ is the

set of all threads $\xi' = \{t'_\alpha\}$ for which $t'_{\alpha_0} \subseteq t_{\alpha_0}$. The set $\hat{S}$ with this topology is a T_0 -space, called the **full limit** of the spectrum S . In S lie the subspaces $\tilde{S}$ and $\dot{S}$ of $\hat{S}$, respectively, called the **upper** and, respectively, **lower limits** of the spectrum S , consisting respectively of the maximal and minimal threads; $\tilde{S}$ and $\dot{S}$ are T_1 -spaces. The question of their nonemptiness is not raised for the present.

A set of simplexes is called **finite** if all its elements are finite-dimensional. Leaving in each complex K_α only the finite-dimensional simplexes belonging to it, we obtain a finite complex N_α —the **finite weakening** of the complex K_α . The projection spectrum $S^\varphi = \{N_\alpha, \omega_\alpha^{\alpha'}\}$ is called the **finite weakening** of the spectrum $S = \{K_\alpha, \omega_\alpha^{\alpha'}\}$.

If α is a covering of a space X , then, along with its finite, classical nerve N_α , we consider the “large” nerve K_α : to any centered set (of any cardinality) of elements of the covering α there corresponds, by definition, a simplex of the large nerve K_α . The covering is called finite if $N_\alpha = K_\alpha$, i.e., if every centered system of elements of the covering α is finite.

Let now, in the space X , as always, a directed family $\mathfrak{z} = \{\alpha\}$ of its decompositions be given. Then the spectrum $S_{\mathfrak{z}} = \{K_\alpha, \omega_\alpha^{\alpha'}\}$ is defined, consisting of the large nerves of these decompositions with the natural projections. For brevity we shall call this spectrum the $\mathfrak{z}$ -spectrum. Its finite weakening will be called the finite $\mathfrak{z}$ -spectrum. In particular, the spectrum S_ξ is denoted by $S_\xi X$ and is called the large spectrum of the space X ; its finite weakening is the Pontryagin spectrum S_X .

Theorem 4. *For any family $\mathfrak{z}$ of decompositions there exist natural** homeomorphisms*

$$\hat{S}_{\mathfrak{z}} = \Omega_{\mathfrak{z}}; \quad \tilde{S}_{\mathfrak{z}} = \omega_{\mathfrak{z}}; \quad \dot{S}_{\mathfrak{z}} = \dot{\omega}_{\mathfrak{z}} \quad \text{and} \quad \hat{S}_{\mathfrak{z}}^\varphi = \Omega_{\mathfrak{z}}^\tau; \quad \tilde{S}_{\mathfrak{z}}^\varphi = \omega_{\mathfrak{z}}^\varphi; \quad \dot{S}_{\mathfrak{z}}^\varphi = \dot{\omega}_{\mathfrak{z}}^\varphi.$$

Remark 2. For $\mathfrak{z} = \mathfrak{z}_X$, theorem 4 contains, as a special case, theorem 1 of [1].

Theorem 5. *Let $\mathfrak{z}$ be a rich conservative family in the space X . Then the upper limit of the $\mathfrak{z}$ -spectrum $S_{\mathfrak{z}}$ is naturally homeomorphic** to the space X if and only if X is a $\mathfrak{z}$ -complete T_ξ -space.*

A consequence of this theorem is:

Theorem 1 $_\xi$. *For ξ -complete T_ξ -spaces and only for them we have*

$$X = \tilde{S}_\xi X.$$

Finally, for $\mathfrak{z} = \mathfrak{z}_X$ theorem 5 becomes theorem 2 of [1]. For finite $\mathfrak{z}$ -spectra a theorem analogous to theorem 5 is valid.

* If a given simplex $T \subseteq E$ is a finite set of vertices, then the number of its vertices diminished by 1 is, as usual, called its dimension; simplexes consisting of an infinite set of vertices are called infinite-dimensional.

** “Natural” means that the homeomorphism arises from the natural mapping $\Xi : \Omega_{\mathfrak{z}} \rightarrow S_{\mathfrak{z}}$, which assigns to the $\mathfrak{z}$ -system ξ the thread $\Xi(\xi) = \{t_\alpha\}$, where $t_\alpha = \xi \cap \alpha$.

*** That is, homeomorphic under the natural mapping $x \rightarrow \{t_\alpha(x)\}$, where $t_\alpha(x) \subseteq \alpha$ is the set of all elements of the covering α containing the point x .

4. Extensions

Theorem 6. If $\mathfrak{z}$ is a rich conservative family in a space X , then $\hat{S}$ is a semiregular T_0 -extension of the space X . If, moreover, X is a T_λ -space, then the extension of the space X is already a T_1 -space $S_{\mathfrak{z}}$.

Let $\bar{X}$ be an extension of the space X ; denote by $\bar{A}$ the closure in $\bar{X}$ of the set $A \in \mathfrak{A}_X$. We obtain a one-to-one correspondence $A \leftrightarrow \bar{A}$ between $\mathfrak{A}_X$ and $\mathfrak{A}_{\bar{X}}$, and between all decompositions $\alpha = \{A, A^\alpha\}$ of the space X and all decompositions $\bar{\alpha} = \{\bar{A}, \bar{A}^\alpha\}$ of the space $\bar{X}$, so that to each family of decompositions $\mathfrak{z} = \{\alpha\}$ of the space X there corresponds a family of decompositions $\bar{\mathfrak{z}} = \{\bar{\alpha}\}$ of the extension $\bar{X}$.

Theorem 7. If X is a semiregular space and in it a rich conservative family $\mathfrak{z}$ is given, then, putting $\bar{X} = \hat{S}_{\mathfrak{z}} = \Omega_{\mathfrak{z}}$, we have, for the family $\bar{\mathfrak{z}}$ in the space $\bar{X}$, the natural homeomorphisms $\hat{S}_{\bar{\mathfrak{z}}} = \hat{S}_{\mathfrak{z}}$, $\tilde{S}_{\bar{\mathfrak{z}}} = \tilde{S}_{\mathfrak{z}}$, $\dot{S}_{\bar{\mathfrak{z}}} = \dot{S}_{\mathfrak{z}}$ and, consequently, $\Omega_{\bar{\mathfrak{z}}} = \Omega_{\mathfrak{z}}$, $\omega_{\bar{\mathfrak{z}}} = \omega_{\mathfrak{z}}$, $\dot{\omega}_{\bar{\mathfrak{z}}} = \dot{\omega}_{\mathfrak{z}}$. If X is a T_λ -space, then in this formulation, instead of $\bar{X} = \hat{S}_{\mathfrak{z}} = \Omega_{\mathfrak{z}}$, one may put $\bar{X} = \tilde{S}_{\mathfrak{z}} = \omega_{\mathfrak{z}}$.

For $\mathfrak{z} = \zeta_X$ one obtains

Theorem 8. Let X be a T_λ -space. Then the space $\omega_\zeta X = \tilde{S}_\zeta X$ is a ζ -complete T_ζ -space which is an extension of the space X ; moreover $\omega_\zeta \omega_\zeta X = \omega_\zeta X$, so that $\omega_\zeta X$ coincides with the upper limit of its large spectrum $S_\zeta \omega_\zeta X$ (a spectrum isomorphic to $S_\zeta X$).

For $\mathfrak{z} = \chi$ we see that, for a T_λ -space X , the space $\omega_\chi X$ is a bicomact T_ζ -extension of the space X , i.e., we obtain Theorem 4 of [1].

Theorems 6 and 7 are consequences of more general necessary and sufficient conditions for the natural mapping $X \rightarrow \hat{S}_{\mathfrak{z}}$ to be an embedding.

I express my heartfelt gratitude to P. S. Aleksandrov, under whose supervision this work was carried out.

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Received
5 XI 1968

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