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Abstract

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MATHEMATICS

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HOMOGENEOUS LOCALLY SYMMETRIC DOMAINS IN CONFORMAL SPACE

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1°. Let us consider a homogeneous space $M = \mathcal{G}'/\mathcal{H}'$, where $\mathcal{H}'$ is the stationary subgroup of a point x_0 . Suppose that a subgroup $\mathcal{G} \subset \mathcal{G}'$ is transitive in some neighborhood of this point; then the orbit of this subgroup $M = \mathcal{G}x_0$, in turn, may be regarded as a homogeneous space $\mathcal{G}/\mathcal{H}$, where $\mathcal{H} = \mathcal{G} \cap \mathcal{H}'$, which is called a **homogeneous domain** in M' .

If the connected subgroup $\mathcal{G}$ preserves in M an affine connection symmetric in the sense of É. Cartan, and if the geodesic symmetry in a neighborhood of the point x_0 preserves the Lie algebra of vector fields of this subgroup, then the domain M is called **homogeneous locally symmetric**. The Lie subalgebra G of such a subgroup $\mathcal{G}$ will also be called **symmetric**.

Let G' and H' denote the Lie algebras of the groups $\mathcal{G}'$ and $\mathcal{H}'$, and let us consider all possible subspaces $E \subset G'$ satisfying the conditions

$$\dim E = \dim M', \quad E \cap H' = \{0\}; \quad (1)$$

$$[E, E] \subset H'; \quad [[E, E], E] \subset E. \quad (2)$$

We shall call any such subspace a **defining plane**. From É. Cartan's conditions characterizing symmetric spaces with an affine connection, it follows, in the case of a conformal space, that a subalgebra $G \subset G'$ is symmetric if and only if

$$G = E + H, \quad (3)$$

where E is some defining plane, $H = G \cap H'$, and $[H, E] \subset E$. Let $H_{\min}$ denote the subalgebra which is the linear span of $[E, E]$, and let $H_{\max}$ be the intersection of the normalizer of E in G' with H' . Obviously,

$$H_{\min} \subset H \subset H_{\max}.$$

Conversely, if E is an arbitrary defining plane and H is a subalgebra satisfying the last condition, then $E + H$ is a symmetric subalgebra.

Thus, the problem of finding all homogeneous locally symmetric domains in M' is reduced essentially to finding all defining planes in the algebra G' . Obviously, the latter problem is solved simultaneously for all homogeneous spaces locally isomorphic to M' .

2°. An important example of a homogeneous symmetric domain in conformal space is the conformal model of Lobachevsky space. In the present paper we consider the generalized conformal space ${}_pC^q$ —henceforth we shall call it simply **conformal**. It is the pseudo-Euclidean space ${}_pE^q$ with scalar square

$$(\mathbf{x}, \mathbf{x}) = - \sum_{i=1}^p x_i^2 + \sum_{i=p+1}^n x_i^2,$$

($n = p + q$), completed in a certain special way by “infinitely

“remote” points, with the group of conformal transformations acting on it (preserving angles in the sense of the form $(\bar{x}, \bar{x})$). Usually the conformal space is obtained in this way starting from ${}_0E^n$.

A model of the space ${}_pC^q$ is the hyperboloid

$$(\mathbf{x}, \mathbf{x}) + x_{n+1}^2 = 1$$

in ${}_pE^{q+1}$, completed in the usual way by its infinitely remote points. Let us denote the vectors of an orthonormal basis of ${}_pE^{q+1}$ by $e_1, e_2, \dots, e_{n+1}$, with $(e_{n+1}, e_{n+1}) = 1$. Stereographic projection from the end of the vector e_{n+1} onto the orthogonal subspace ${}_pE^q$ introduces in ${}_pC^q$ a local coordinate system T . Similarly, projecting from the point $-e_{n+1}$, we obtain the system T_0 . Finally, introduce systems T_i (for those values $i \leq n$ for which $(e_i, e_i) = 1$) by first performing a rotation which sends e_i to e_{n+1} , leaving the other basis vectors fixed, and then a stereographic projection from the point e_{n+1} . The indicated systems cover all points of the space.

Note that ${}_pC^q$ is doubly covered by the product $S^p \times S^q$ (we denote by S^n the sphere in ${}_0E^{n+1}$).

The Lie algebra of the group of conformal transformations of the space ${}_pE^q$, considered as the Lie algebra of vector fields, is the linear envelope of the following operators defining vector fields: all vectors from ${}_pE^q$, i.e. constant operators; all linear operators skew-symmetric with respect to the form $(\mathbf{x}, \mathbf{x})$; the identity operator, and all quadratic operators K_a of the form

$$\mathbf{x} \rightarrow 2(\mathbf{a}, \mathbf{x})\mathbf{x} - (\mathbf{x}, \mathbf{x})\mathbf{a},$$

where $\mathbf{a} \in {}_pE^q$.

Note that the transformations Adh , where $h \in \mathcal{H}'$, preserve the set of determining planes. The problem of finding all determining planes in the algebra G' is solved by the following theorem.

Theorem 1. *An arbitrary determining plane in the Lie algebra of the group of transformations of the spaces ${}_pC^q$ is brought by some transformation Adh , where $h \in \mathcal{H}'$, to the form $\{\mathbf{a} + K_{R\mathbf{a}} \mid \mathbf{a} \in {}_pE^q\}$, where R is a linear mapping of ${}_pE^q$ into itself, symmetric with respect to the form $(\mathbf{x}, \mathbf{x})$ and satisfying the condition $R^2 = \alpha I$, in which α is equal to 1, 0, or -1 , and I is the identity mapping. Conversely, every plane of the indicated form is determining.*

We shall call a homogeneous locally symmetric domain $M = \mathcal{G}x_0$ a **domain of the first, second, or third type**, respectively, if some transformation Adh , $h \in \mathcal{H}'$, brings its determining plane (i.e. the plane E entering into the decomposition (3)) to the form $\{\mathbf{a} + K_{R\mathbf{a}} \mid \mathbf{a} \in {}_pE^q\}$, where R^2 is I , 0, or $-I$, respectively. The domain hM will be called a **domain reduced to canonical form**. For domains of the first and third type the subalgebras $H_{\min}$ and $H_{\max}$ coincide; for domains of the second type they are different.

3°. Denote by ${}_pS^q$ the homogeneous space $SO(p, q+1)/SO(p, q)$. In the formulation of the theorem given below, products of the form ${}_pS^q \times {}_rS^s$ are used. For generality of notation it is permitted, for example, that $p+q=0$; then necessarily $r+s>0$, and the product is identified with ${}_rS^s$.

Consider an affine space A of dimension $n = 2(l+m) + r + s$, fix in it some basis, and introduce a scalar product with values in the algebra of dual numbers by means of the formula

$$(\mathbf{x}, \mathbf{x})_\varepsilon = - \sum_{i=1}^l x_i^2 + \sum_{i=l+1}^{l+m} x_i^2 + \varepsilon \left(-2 \sum_{i=l+m+1}^{2l+m} x_i x_{i-l-m} + 2 \sum_{i=2l+m+1}^{2l+2m} x_i x_{i-l-m} - \sum_{i=2l+2m+1}^{n-s} x_i^2 + \sum_{i=n-s+1}^n x_i^2 \right),$$

where ε is the dual unit, $\varepsilon^2 = 0$ *). Assuming that $m > 0$, denote by ${}_{l,r}S^{m-1,s}$ the set of vectors such that $(x, x)_\varepsilon = 1$. Consider the group $P(l, m, r, s)$ of all linear transformations in A , each of which, acting on vectors x and y from A , preserves the real part of their product and multiplies the coefficient of the dual part by some positive number independent of x and y . It turns out that, as a homogeneous space, the "sphere" ${}_{l,r}S^{m,s}$ is $P(l, m+1, r, s)/P(l, m, r, s)$. Finally, denote by $S^n(C)$ the homogeneous space $SO(n+1, C)/SO(n, C)$. Note that, when considering the conformal space ${}_pC^q$, one may assume $p \leq q$.

Theorem 2. *In the conformal space ${}_pC^q$, any domain of the first type is locally isomorphic to one of the homogeneous spaces ${}_rS^s \times {}_{q-s}S^{p-r}$, where $0 \leq r \leq p$; $0 \leq s \leq q$. Any domain of the second type is locally isomorphic to one of the spaces ${}_{l,p-l-m}S^{m,q-l-m}$, where $l, m \geq 0$; $l+m \leq p$ **. If $p = q$, then there also exist domains of the third type, all of which are locally isomorphic

to $S^p(C)$. Conversely, in ${}_pC^q$ there exist domains locally isomorphic to all the indicated spaces.*

4°. If one assumes that the subgroup $\mathfrak{G}$ is chosen connected, then the domain M can be defined as a component of the domain of transitivity of the subgroup $\mathfrak{G}$ containing the point x_0 . By the **complete boundary** we shall mean the set of points at which $\mathfrak{G}$ loses transitivity. In distinguishing the complete boundary, the main role is played by the mapping R mentioned in Theorem 1.

We shall call a curve in conformal space a “**straight line**” if, in one of the local coordinate systems indicated earlier, it is represented by a straight line. Each “straight line” has a unique limiting point, which we also include among the points of this “straight line.”

Consider a domain of the first type, reduced to canonical form. Denote by R_+ and R_- the eigenspaces of the mapping R corresponding to the eigenvalues 1 and -1 . The spheres of radii 1, -1 , and 0 in ${}_pE^q$ will be denoted respectively by S_1 , S_{-1} , and K . Next put $S_+ = S_{-1} \cap R_+$, $S_- = S_1 \cap R_-$, and also $K_+ = K \cap R_+ \setminus \{O\}$ and $K_- = K \cap R_- \setminus \{O\}$. Finally, denote by $\Gamma_+(\Gamma_-)$ the set consisting of $S_+(S_-)$ and the limiting points of the “straight lines” $\{\bar{a}t\}$, where $\bar{a} \in K_+(\bar{a} \in K_-)$.

Theorem 3. *The complete boundary in the case of a domain of the first type reduced to canonical form is the set Γ , consisting of all “straight lines” joining points of Γ_+ with points of Γ_- , if both these sets are nonempty; if one of them is empty, then Γ coincides with the other.*

In ordinary conformal space ${}_0C^n$, which may be regarded as the sphere S^n , it follows from Theorem 2 that there exist only domains of the first type, isomorphic to $L^k \times S^{n-k}$, $0 \leq k \leq n$, where L^k is k -dimensional Lobachevskii space (note that for $k = 0$ the boundary is empty, while for $k > 0$ its dimension is $k - 1$), and also domains of the second type, isomorphic to the space ${}_0E^n$ with the group of similarities acting on it. We indicate the boundaries of all domains directly on S^n . The intersection of S^n and a $k + 1$ -dimensional plane ($0 \leq k \leq n - 1$) will be called a k -sphere in S^n .

1. Any k -sphere in S^n is the boundary of some homogeneous symmetric domain, and all nonempty boundaries are exhausted in this way. If the given k -sphere degenerates to a point, then the domain is of the second type; otherwise the domain is of the first type.
2. The maximal connected subgroup of the conformal group preserving a given k -sphere in S^n is the subgroup generating the homogeneous symmetric domain whose boundary is this k -sphere.

* If $r + s = 0$, then in this way we obtain the pseudo-Euclidean space ${}_lE^m$ over the algebra of dual numbers.

** As the transformation group of a domain of the second type, here and below one has in mind the subgroup of the group $\mathfrak{G}'$ corresponding to the subalgebra $E + H_{\max}$.

Theorem 3₂. The complete boundary in the case of a domain of the second type, reduced to canonical form, consists of the surface given in the local coordinate system T_0 by the equation

$$(\mathbf{x}, \mathbf{x})^2 + 2(R\mathbf{x}, \mathbf{x}) = 0,$$

and of all its limit points.

Theorem 3₃. The complete boundary in the case of a domain of the third type, reduced to canonical form, consists of the surface given in the local coordinate system T by the equation

$$(\mathbf{x}, \mathbf{x})^2 - 2(R\mathbf{x}, \mathbf{x}) = 1,$$

and of all its limit points.

The mapping $\mathbf{x} \rightarrow -\mathbf{x}$ in a neighborhood of the point $x_0 = 0 \in T$ coincides with the geodesic symmetry for any domain reduced to canonical form. On the other hand, this mapping extends to the whole space and, as follows from Theorem 3, preserves the domains reduced to canonical form. Hence it follows that all the domains found are not only locally, but also globally symmetric.

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Note: Figure translations are in progress. See original paper for figures.

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