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RESONANCE THEOREMS FOR FUNCTIONAL SERIES AND SEQUENCES

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Abstract

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MATHEMATICS

E. M. NIKISHIN

RESONANCE THEOREMS FOR FUNCTIONAL SERIES AND SEQUENCES

(Presented by Academician A. N. Kolmogorov on 14 XI 1968)

The following theorem of E. Landau is known:

If the series $\sum_{n=1}^{\infty} a_n b_n$ converges for all $\{b_n\} \in l_p$, then $\{a_n\} \in l_q$ ($1/p + 1/q = 1$).

The situation changes sharply if, instead of a numerical sequence $\{a_n\}$, one takes a functional one and convergence is understood as convergence almost everywhere. The set of divergence in this case (it has measure 0) depends on the chosen coefficients, i.e., it “moves” when they are changed. Landau’s theorem proves powerless to help us in clarifying the metric properties of a functional sequence. Nevertheless, many questions in the theory of Fourier series correspond to such a situation, when it is necessary to establish what metric restrictions on a system of functions $\{f_n(x)\}$ imply convergence of the series

$$\sum_{n=1}^{\infty} b_n f_n(x), \quad (x \in [0, 1]), \quad (1)$$

if $\{b_n\}$ are chosen from some set B . Here convergence may be understood in different ways: as absolute convergence almost everywhere, as convergence in measure, or simply as convergence almost everywhere. The present paper is devoted to the solution of certain problems from this circle of questions.

Let B be a Banach space whose elements are numerical sequences, and let the algebraic operations be defined in the natural way. Suppose the following conditions are satisfied:

- 1) if $\bar{a} = \{a_k\} \in B$, then $\bar{a}_n = \{a_1, a_2, \dots, a_n, 0, 0, \dots\} \in B$ and $\|\bar{a}_n\| \leq \|\bar{a}\|$;
- 2) if $\bar{a} \in B$ and $\bar{a}_{mn} = \{0, 0, \dots, 0, a_m, a_{m+1}, \dots, a_n, 0, 0, \dots\}$, then

$$\lim_{m, n \rightarrow \infty} \|\bar{a}_{mn}\| = 0.$$

Definition 1. A system of measurable functions $\{f_n(x)\}$ is called a **system of absolute convergence** for B if the series (1) converges absolutely almost everywhere whenever $\{b_n\} \in B$.

Definition 2. A system of measurable functions $\{f_n(x)\}$ is called a **system of convergence in measure** for B if the series (1) converges in measure whenever $\{b_n\} \in B$.

Definition 3. A system of measurable functions $\{f_n(x)\}$ is called a **system of convergence** for B if the series (1) converges almost everywhere on $[0, 1]$ whenever $\{b_n\} \in B$.

Theorem 1. In order that a system of functions $\{f_n(x)\}$ be a system of absolute convergence for l_p ($p \geq 1$), it is necessary and sufficient that for every $\varepsilon > 0$ there exist a set $E_\varepsilon \subset [0, 1]$ with $mE_\varepsilon \geq 1 - \varepsilon$ such that

$$\sum_{n=1}^{\infty} \left(\int_{E_\varepsilon} |f_n(x)| dx \right)^q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{for } p > 1;$$

$$\sup_{n \geq 1} \int_{E_\varepsilon} |f_n(x)| dx < \infty \quad \text{for } p = 1.$$

For other theorems concerning systems of absolute convergence, see $(^1, ^2)$.

Theorem 2. In order that the system $\{f_n(x)\}$ be a system of convergence in measure for B , it is necessary and sufficient that, for any $\varepsilon > 0$ and $0 < \delta < 1$, there exist a set $E_{\varepsilon, \delta}$ with $mE_{\varepsilon, \delta} \geq 1 - \varepsilon$ and a constant $C_{\varepsilon, \delta}$ such that

$$\int_{E_{\varepsilon, \delta}} \left| \sum_{k=1}^N a_k f_k(x) \right|^{1-\delta} dx \leq C_{\varepsilon, \delta} \|\bar{a}\|^{1-\delta}$$

for every $N \geq 1$ and every $\bar{a} = \{a_k\} \in B$.

Theorem 3. In order that the system $\{f_n(x)\}$ be a system of convergence for B , it is necessary and sufficient that, for any $\varepsilon > 0$ and $0 < \delta < 1$, there exist a set $E_{\varepsilon, \delta}$ (with $mE_{\varepsilon, \delta} \geq 1 - \varepsilon$) and a constant $C_{\varepsilon, \delta} > 0$ such that

$$\int_{E_{\varepsilon, \delta}} \sup_{\mu, \nu \geq 1} \left| \sum_{k=\mu}^{\nu} a_k f_k(x) \right|^{1-\delta} dx \leq C_{\varepsilon, \delta} \|\bar{a}\|^{1-\delta}$$

for every sequence $\bar{a} = \{a_k\} \in B$.

We note that, in proving the necessity in Theorems 2 and 3, for B only the fulfillment of condition 1) is required.

From Theorem 3 it follows that

Theorem 4. In order that the orthonormal system $\{\varphi_n(x)\}$ have the Weyl multiplier for convergence $\omega(n)$, it is necessary and sufficient that, for any $\varepsilon > 0$ and $0 < \delta < 1$, there exist a set $E_{\varepsilon, \delta}$ (with $mE_{\varepsilon, \delta} \geq 1 - \varepsilon$) and a constant $C_{\varepsilon, \delta} > 0$ such that

$$\int_{E_{\varepsilon, \delta}} \sup_{\mu, \nu \geq 1} \left| \sum_{k=\mu}^{\nu} \frac{a_k \varphi_k(x)}{\sqrt{\omega(k)}} \right|^{1-\delta} dx \leq C_{\varepsilon, \delta} \left[\left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2} \right]^{1-\delta}$$

for every sequence $\{a_k\} \in l_2$.

It follows from the results formulated below that Theorems 2 and 3 are, in general, unimprovable (δ cannot be taken equal to 0). The analogous question for Theorem 4 remains open.

Let us formulate several more theorems of the same type.

Definition 4. We shall say that the system $\{f_n(x)\}$ has **property A** if, for every $\{\xi_n\} \in l_1$, it is true that $\lim_{n \rightarrow \infty} \xi_n f_n(x) = 0$ almost everywhere on $[0, 1]$.

Theorem 5. In order that the system $\{f_n(x)\}$ have property A, it is necessary and sufficient that, for every $\varepsilon > 0$, there exist a set E_ε (with $mE_\varepsilon \geq 1 - \varepsilon$) and a constant $c_\varepsilon > 0$ such that, for all $y > 0$,

$$m\{x \in E_\varepsilon; |f_n(x)| \geq y\} \leq c_\varepsilon / y.$$

It is interesting that property A is equivalent to the following: the series $\sum_{n=1}^{\infty} |\xi_n f_n(x)|^p$ (here $p > 1$ is arbitrary) converges almost everywhere as soon as $\{\xi_n\} \in l_1$.

It is obvious that if the system $\{f_n(x)\}$ is a system of convergence for l_1 , then $\{f_n(x)\}$ has property A. For systems having property A, one can prove the following theorem:

Theorem 6. If $\{f_n(x)\}$ is a system having property A, then for every $\varepsilon > 0$ one can find a set $E_\varepsilon \subseteq [0, 1]$ (with $mE_\varepsilon \geq 1 - \varepsilon$)

such that

$$\int_{E_\varepsilon} |f_n(x)| dx = O(\ln n).$$

Nevertheless, one can give an example of a convergence system for l_1 such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{\ln n} \int_E |f_n(x)| dx > 0$$

for any set E (with $mE > 0$).

In conclusion we note that Theorems 2 and 3 admit a strengthening for concrete systems $\{f_n(x)\}$. Let us indicate, for example, the following theorem on systems of the form $\{\varphi(nx)\}$ (or $\{\varphi(2^n x)\}$), where $\varphi(x)$ is a certain periodic (with period 1) function belonging to $L_2[0, 1]$ and with mean equal to 0, i.e.

$$\int_0^1 \varphi(x) dx = 0.$$

Theorem 7. *In order that the system $\{\varphi(nx)\}$ (or $\{\varphi(2^n x)\}$) be a convergence system with respect to measure for l_2 , it is necessary and sufficient that, for every $1 \leq p < 2$, the inequality*

$$\int_0^1 \left| \sum_{n=1}^N a_n \varphi(nx) \right|^p dx \leq C_p \left(\sum_{n=1}^N a_n^2 \right)^{p/2}$$

$$\left(\text{or } \int_0^1 \left| \sum_{n=1}^N a_n \varphi(2^n x) \right|^p dx \leq C_p \left(\sum_{n=1}^N a_n^2 \right)^{p/2} \right)$$

for all $N \geq 1$ and arbitrary numbers $\{a_n\}$.

Using Theorem 7, one can prove the following theorem:

Theorem 8. *There exists a continuous function $\varphi(x)$ ($\varphi(x+1) = \varphi(x)$, $\int_0^1 \varphi(x) dx = 0$) and numbers $\{a_k\} \in l_2$ such that the series $\sum_{k=1}^{\infty} a_k \varphi(2^k x)$ diverges almost everywhere and does not converge in measure on $[0, 1]$.*

This theorem gives an answer to a question of P. L. Ul'yanov, formulated by him in ⁽³⁾. We note that the first example of this sort was constructed by Erdős (see ⁽⁴⁾), but in his example the function $\varphi(x)$ was essentially unbounded. This example was strengthened by V. F. Gaposhkin; the function $\varphi(x)$ in his example possessed a quadratic modulus of continuity $\omega_2(\delta, \varphi) = O(\ln 1/\delta)^{-1/2}$ (see ⁽⁵⁾); however, in his example as well the function $\varphi(x)$ was essentially unbounded.

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Note: Figure translations are in progress. See original paper for figures.

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