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Abstract

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MATHEMATICS

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APPROXIMATE SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES

(Presented by Academician L. S. Pontryagin on 24 VII 1969)

A system of equations in vector form is considered:

$$\varepsilon dx/dt = f(x, y), \quad dy/dt = g(x, y) \quad (1)$$

$$(x = (x^1, \dots, x^k); \quad y = (y^1, \dots, y^l)).$$

The functions $f(x, y) = \{f^1(x, y), \dots, f^k(x, y)\}$ and $g(x, y) = \{g^1(x, y), \dots, g^l(x, y)\}$ are defined and several times differentiable with respect to their arguments in some domain of the space of variables (x, y) ; ε is a small positive parameter.

In the present article we consider cases in which the system of fast motions

$$dx/d\tau = f(x, y) \quad (\tau = t/\varepsilon) \quad (2)$$

(here y is a constant vector) has either a degenerate equilibrium position or a degenerate limit cycle.

Suppose first that, for some $y = y_0$, system (2) has an equilibrium position $x = x_0$, and that the matrix $A = \|\partial f^i/\partial x^j\|$ at the point (x_0, y_0) has a zero eigenvalue of multiplicity one, while all the remaining eigenvalues have negative real parts.

Introduce the eigenvectors $m = (m^1, \dots, m^k)$ and $n = (n_1, \dots, n_k)$ of the matrix A and of the transposed matrix A' , corresponding to the zero eigenvalue and normalized so that $m^\alpha n_\alpha = 1$. We require the fulfillment of the additional conditions:

$$p = n_\delta f_{x^\alpha x^\beta}^\delta(x_0, y_0) m^\alpha m^\beta \neq 0; \quad n_\delta f_{y^\mu}^\delta(x_0, y_0) = 0 \quad (\mu = 1, \dots, l).$$

If, say, $m^1 \neq 0$, $n_1 \neq 0$, then the equations $f^2(x, y) = \dots = f^k(x, y) = \det \|\partial f^i / \partial x^j\| = 0$ determine $x^1, \dots, x^k$ as functions of y ; substituting these functions into $f^1(x, y)$, we obtain a certain function $H(y)$ with the following properties: $H(y_0) = 0$, $\partial H(y_0) / \partial y^i = 0$ ($i = 1, \dots, l$).

We impose on the function $H(y)$ such conditions under which $H(y) = 0$ determines, in some neighborhood of the point y_0 , a smooth surface Π , say a surface of the form $y^1 = h(y^2, \dots, y^l)$. From the form of the function $H(y)$ it follows that it preserves a constant sign at points not belonging to the surface Π . If the signs of the functions $H(y)$ and p are opposite, then system (2) has one degenerate equilibrium position for y lying on the surface Π , and two nondegenerate equilibrium positions $x = \psi_1(y)$ and $x = \psi_2(y)$ for y not lying on Π ; moreover, if y lies on one side of Π (for example, $y^1 < h(y^2, \dots, y^l)$), then $\psi_1(y)$ is a stable equilibrium position, and if y lies on the other side of Π ($y^1 > h(y^2, \dots, y^l)$), then $\psi_2(y)$ is a stable equilibrium position.

We shall assume that the following additional conditions are satisfied:

1. The expressions $n_\delta \psi_i^\gamma g^\alpha \big|_{\substack{x=x_0 \\ y=y_0}} \neq 0$ ($i = 1, 2$), and if they have the same sign, then it coincides with the sign of p .
2. The vector $g(x_0, y_0)$ is not tangent to the surface Π , for example,

$$g^1 - g^\alpha \partial h / \partial y^\alpha \big|_{\substack{x=x_0 \\ y=y_0}} > 0 \quad (\alpha = 2, \dots, l). \quad (3)$$

Let the point y_1 be sufficiently close to y_0 and satisfy the inequality $y_1^1 < h(y_1^2, \dots, y_1^l)$. Denote by $Y_1(t)$ the solution of the system $dy/dt = g(\psi_1(y), y)$ with initial condition $Y_1(t_1) = y_1$. It follows from condition (3) that the solution $Y_1(t)$ at some time $t = t_2$ falls on the surface Π .

Denote by $Y_2(t)$ the solution of the system $dy/dt = g(\psi_2(y), y)$ with initial condition $Y_2(t_2) = Y_1(t_2)$, defined on some interval $[t_2, t_3]$.

Theorem 1. *If the solution $x(t, \varepsilon), y(t, \varepsilon)$ of system (1) satisfies the condition $|x(t_1, \varepsilon) - \psi_1(y_1)| = O(\varepsilon)$, $|y(t_1, \varepsilon) - y_1| = O(\varepsilon)$, then on the interval $[t_1, t_3]$ $x(t, \varepsilon) \rightarrow X(t)$, $y(t, \varepsilon) \rightarrow Y(t)$ as $\varepsilon \rightarrow 0$, where*

$$X(t) = \begin{cases} \psi_1[Y_1(t)], & t_1 \leq t \leq t_2, \\ \psi_2[Y_2(t)], & t_2 < t \leq t_3; \end{cases}$$

$$Y(t) = \begin{cases} Y_1(t), & t_1 \leq t \leq t_2, \\ Y_2(t), & t_2 < t \leq t_3. \end{cases}$$

Let us now consider the case when system (2), for some $y = y_0$, has a degenerate limit cycle $x = \varphi_0(\tau)$ of period T_0 . For system (2) we construct an analogue of the successor function, which here will be a mapping of the $(k-1)$ -dimensional

space of variables $u^1, \dots, u^{k-1}$ into itself: $v = \chi(u, y)$. We shall assume that the matrix

$$M = \|\partial \chi^i(u_0; y_0) / \partial u^j\| \quad (i, j = 1, \dots, k-1)$$

(u_0 is the fixed point of the mapping corresponding to the cycle $\varphi_0(\tau)$) has an eigenvalue equal to unity, of multiplicity one, while all the remaining eigenvalues are less than unity in modulus. Introduce the eigenvectors $m = (m^1, \dots, m^{k-1})$ and $n = (n_1, \dots, n_{k-1})$ of the matrix M and of the transposed matrix M' , corresponding to the unit eigenvalue and normalized so that $m^\alpha n_\alpha = 1$. On the function $\chi(u, y)$ impose the nondegeneracy conditions:

$$p = n_\delta \chi_{u^\alpha u^\beta}^\delta(u_0, y_0) m^\alpha m^\beta \neq 0,$$

$$q = \{q_1, \dots, q_l\} \neq 0, \quad \text{where} \quad q_\mu = n_\delta \chi_{y^\mu}^\delta(u_0, y_0).$$

It follows from these conditions that the system

$$\chi(u, y) = u, \quad \det \|\partial \chi(u, y) / \partial y - E\| = 0$$

defines in the space y , in some neighborhood of the point y_0 , a smooth surface Π passing through the point y_0 and possessing the following property: if y lies on one side of Π , then system (2) has two limit cycles, one of which $x = \varphi(\tau, y)$ is nondegenerate stable with period $T(y)$; if y belongs to Π , then system (2) has one degenerate limit cycle; and if y lies on the other side of the surface Π , then system (2) has no limit cycles.

Let us introduce for consideration the averaged system

$$d\bar{y}/dt = \bar{g}(\bar{y}), \tag{4}$$

where

$$\bar{g}(\bar{y}) = \frac{1}{T(\bar{y})} \int_0^{T(\bar{y})} g(\varphi(\theta, \bar{y}), \bar{y}) d\theta; \quad \bar{g}(y_0) = \frac{1}{T_0} \int_0^{T_0} g(\varphi_0(\theta), y_0) d\theta.$$

We shall assume that the vector $\bar{g}(y_0)$ is not tangent to the surface Π and is directed toward that side of the space y where those values of y are located for which system (2) has no limit cycles.

Let the point y_1 be sufficiently close to y_0 , and let system (2) for $y = y_1$ have two nondegenerate limit cycles. Denote by $\bar{y}(t)$ the solution of system (4) with the initial condition $\bar{y}(t_1) = y_1$. From the condition imposed on the vector $\bar{g}(y_0)$ it follows that at some moment of time $t = t_2$ the solution $\bar{y}(t)$ reaches the surface Π . Without loss of generality, we may assume that $\bar{y}(t_2) = y_0$.

Theorem 2. *If the solution $x(t, \varepsilon), y(t, \varepsilon)$ of system (1) satisfies the condition*

$$|x(t_1, \varepsilon) - \varphi(t_1/\varepsilon, y_1)| = O(\varepsilon), \quad |y(t_1, \varepsilon) - y_1| = O(\varepsilon),$$

then there exists a function $\theta(t, \varepsilon)$, smoothly depending on t , such that as $\varepsilon \rightarrow 0$

$$|\varepsilon d\theta/dt - 1| \rightarrow 0, \quad |y(t, \varepsilon) - \bar{y}(t)| \rightarrow 0,$$

$$|x(t, \varepsilon) - \varphi(\theta(t, \varepsilon), \bar{y}(t))| \rightarrow 0, \quad t_1 \leq t \leq t_2,$$

and such a $t_3(\varepsilon) > t_2$, where $t_3(\varepsilon) \rightarrow t_2$ as $\varepsilon \rightarrow 0$, that the point $x(t_3(\varepsilon), \varepsilon), y(t_3(\varepsilon), \varepsilon)$ lies outside a small, but ε -independent, neighborhood of the closed curve $x = \varphi_0(\tau), y = y_0$, and moreover $|y(t_3(\varepsilon), \varepsilon) - y_0| \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Finally, consider the case in which the function $\chi(u, y)$ satisfies the conditions $p \neq 0, q = 0$. One may introduce a function $H(y)$, analogous to the function $H(y)$ considered in theorem (1). The equation $H(y) = 0$ defines the surface Π , whose equation can be written in the form $y^1 = h(y^2, \dots, y^l)$. We shall assume that the signs of the function $H(y)$ and of p are opposite, so that for y lying on the surface Π , the mapping $v = \chi(u, y)$ has one fixed point, while system (2) has one degenerate limit cycle $x = \varphi_0(\tau, y)$; whereas if y does not lie on Π , the mapping $v = \chi(u, y)$ has two fixed points $u = \psi_1(y)$ and $u = \psi_2(y)$, and system (2) has two limit cycles. Denote the limit cycles of system (2) corresponding to these fixed points by $x = \varphi_1(\tau, y)$ (stable if $y^1 < h(y^2, \dots, y^l)$) and $x = \varphi_2(\tau, y)$ (stable if $y^1 > h(y^2, \dots, y^l)$). Introduce into consideration the averaged system

$$\frac{d\bar{y}}{dt} = \bar{g}_i(\bar{y}) \quad (i = 1, 2), \quad \text{where} \quad g_i(\bar{y}) = \frac{1}{T_i(\bar{y})} \int_0^{T_i(\bar{y})} g(\varphi_i(\tau, \bar{y}), \bar{y}) d\tau;$$

$i = 1$, if $\bar{y}^1 < h(\bar{y}^2, \dots, \bar{y}^l)$, and $i = 2$, if $\bar{y}^1 > h(\bar{y}^2, \dots, \bar{y}^l)$. $T_i(\bar{y})$ is the period of the cycle $\varphi_i(\tau, \bar{y})$; $i = 0$, if $\bar{y} \in \Pi$.

We shall assume that the expressions

$$n_\delta \psi_{iy^\alpha}^\delta \bar{g}^\alpha|_{y=y_0} \neq 0 \quad (i = 1, 2),$$

and if they have the same sign, then it coincides with the sign of p , and that the vector $\bar{g}(y_0)$ is not tangent to the surface Π , for example,

$$\bar{g}^1 - \bar{g}^\alpha \partial h / \partial y^\alpha|_{y=y_0} > 0 \quad (\alpha = 2, \dots, l). \quad (5)$$

Let the point y_1 be sufficiently close to the point y_0 and satisfy the inequality $y_1^1 < h(y_1^2, \dots, y_1^l)$. Denote by $Y_1(t)$ the solution of the averaged system for $i = 1$ with the initial condition $Y_1(t_1) = y_1$. From condition (5) it follows that the solution $Y_1(t)$, at some moment of time $t = t_2$, reaches the surface Π . Denote by $Y_2(t)$ the solution of the averaged system for $i = 2$ with the initial condition $Y_2(t_2) = Y_1(t_2)$, defined on some interval $[t_2; t_3]$.

Theorem 3. *If the solution $x(t, \varepsilon), y(t, \varepsilon)$ of system (1) satisfies the conditions*

$$|x(t_1, \varepsilon) - \varphi_1(t_1/\varepsilon, y_1)| = O(\varepsilon), \quad |y(t_1, \varepsilon) - y_1| = O(\varepsilon),$$

then there exists a function $\theta(t, \varepsilon)$, smoothly depending on t , such that, as $\varepsilon \rightarrow 0$,

$$|\varepsilon d\theta/dt - 1| \rightarrow 0, \quad |x(t, \varepsilon) - \varphi_i(\theta(t, \varepsilon), Y_i(t))| \rightarrow 0,$$

$$|y(t, \varepsilon) - Y_i(t)| \rightarrow 0,$$

where $i = 1$ for $t_1 \leq t \leq t_2$; $i = 2$ for $t_2 < t \leq t_3$.

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