

ON THE DISTRIBUTION OF EIGENVALUES OF SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

M. G. Gasymov

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Abstract

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MATHEMATICS

M. G. GASIMOV

ON THE DISTRIBUTION OF EIGENVALUES OF SELF-ADJOINT ORDINARY DIFFERENTIAL OPERATORS

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§ 1. Statement of the problem. Consider the formally symmetric differential expression

$$l\left(x, \frac{d}{dx}\right)y = (-1)^n y^{(2n)} + \sum_{j=0}^{n-1} (-1)^j \{p_j(x)y^{(j)}(x)\}^{(j)}, \quad -\infty < x < \infty,$$

and suppose that its minimal extension in $L_2(-\infty, \infty)$ generates a self-adjoint operator L , bounded below by zero. Let the operator L have a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots$ with corresponding normalized eigenfunctions $\varphi_1(x), \varphi_2(x), \dots$. Put

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1.$$

In the present paper we shall find the principal term of the asymptotics of the function $N(\lambda)$ as $\lambda \rightarrow +\infty$ in the case when it is determined by one or several coefficients of the expression $l(x, d/dx)$; here, in contrast to previous works, in our considerations the free term $p_0(x)$ is subordinated to the other coefficients and may even have no effect on the principal term of the asymptotics.

There is a large series of works devoted to the study of the asymptotics of $N(\lambda)$ for various elliptic operators in $L_2(\Omega)$, where Ω is a finite or infinite domain in E_n (for a complete bibliography see ⁽¹⁾).

Up to now the literature has considered the case when the principal term of the asymptotics of $N(\lambda)$ is determined by the principal part and the free coefficient of the elliptic operator. An exception is the work of A. G. Kostyuchenko ⁽²⁾, where all coefficients have the same influence on the asymptotics of $N(\lambda)$. However, in ⁽²⁾, besides conditions on the coefficients, conditions are also imposed on the roots of the corresponding characteristic equation.

§ 2. Conditions on the coefficients $p_j(x)$. In what follows we assume that the functions $p_0(x), \dots, p_{n-1}(x)$ satisfy the following conditions (by $c_1, c_2, \dots$ various positive constants are denoted):

1.

$$c_1(1 + |x|)^{\alpha_j} \leq p_j(x) \leq c_2(1 + |x|)^{\alpha_j}$$

for large x .

2. For $|u - x| \leq |u|/2$,

$$|p_j(u) - p_j(x)| \leq C|x - u|(1 + |x|)^{\alpha_j - 1}.$$

3. For $|x - u| \geq |u|/2 \geq 1$,

$$|p_j(u) - p_j(x)| \leq C|x - u|^{[\alpha_j] + 1}(1 + |u|)^{\alpha_j - [\alpha_j] - 1}.$$

4. Each function $p_j(x)$ has derivatives up to order j , inclusive, and

$$|p_j^{(k)}(x)| \leq C(1 + |x|)^{\alpha_j - k}, \quad k = 1, \dots, j.$$

5. For some positive integer k the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx d\xi}{\{l(x, i\xi) + 1\}^{2k}} \quad (1)$$

converges.

We note that condition 2 follows from condition 4, while condition 5 imposes certain restrictions on the numbers $a_0, a_1, \dots, a_{n-1}$. They are given in Lemma 1. In all other respects the conditions are independent.

Lemma 1. *The integral (1) converges if and only if for at least one j_0 the number a_{j_0} is greater than $2j_0$; moreover, if $j_0 > 0$, then one may put $k = 1$, while if $j_0 = 0$, then $k \geq [1/a_0] + 1$.*

II. Asymptotics of the trace of $(L + \mu)^{-2k}$.

Theorem 1. *Suppose that conditions 1-5 of § 2 are satisfied. Then $(L + \mu)^{-2k}$ is a nuclear operator and, for large positive μ ,*

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda_n + \mu)^{2k}} \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dx d\xi}{\{l(x, i\xi) + \mu\}^{2k}}. \quad (2)$$

Let us briefly outline the proof of the theorem.

Our arguments are similar to those of Titchmarsh ([3], Ch. XVII). In his case, as well as in all cases considered up to now, the kernel of the resolvent $(L_v + \mu)^{-2k}$ of

the operator L_v , which is constructed below, decreases exponentially as $|v| \rightarrow \infty$, and this simplifies the arguments. However, in our case, when the free term $p_0(x)$ is subordinate to the intermediate coefficients, this fact does not hold. More precisely: the kernel of the resolvent $(L_v + \mu)^{-2k}$ has only a power order of decrease as $|v| \rightarrow \infty$. For simplicity, let us assume that the integral (1) converges for $k = 1$.

Denote by L_v the operator generated in $L_2(-\infty, \infty)$ by the differential expression $l(v, d/dx)$, which is obtained from $l(x, d/dx)$ by replacing x by v in the coefficients. Denote by $G(\mu; x, u; v)$ the kernel of the resolvent $(L_v + \mu)^{-1}$. It is known that

$$G(\mu; x, u; v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-u)}}{l(v, i\xi) + \mu} d\xi. \quad (3)$$

Then (see [3], Ch. XVII)

$$\frac{\varphi_n(u)}{\lambda_n + \mu} = \int_{-\infty}^{\infty} G(\mu; x, u; v) \varphi_n(x) dx - \int_{-\infty}^{\infty} \frac{\varphi_n(x)}{\lambda_n + \mu} \left\{ l\left(x, \frac{d}{dx}\right) - l\left(v, \frac{d}{dx}\right) \right\} G(\mu; x, u; v) dx. \quad (4)$$

Put $v = u$ in the last identity; denote the first term on the right-hand side by $a_n(u, \mu)$, and the second by $b_n(u, \mu)$. Squaring both sides and integrating with respect to u over the interval $(-\infty, \infty)$, we obtain as a result

$$\frac{1}{(\lambda_n + \mu)^2} = \int_{-\infty}^{\infty} a_n^2(u, \mu) du + 2 \int_{-\infty}^{\infty} a_n(u, \mu) b_n(u, \mu) + \int_{-\infty}^{\infty} b_n^2(u, \mu) du.$$

From Parseval's equality it follows that

$$\begin{aligned} A(\mu) &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} a_n^2(u, \mu) du = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G^2(\mu; x, u; v)| du dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\xi dx}{\{l(x, i\xi) + \mu\}^2}, \end{aligned} \quad (5)$$

and the last integral exists. Put

$$B(\mu) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} b_n^2(u, \mu) du. \quad (6)$$

Further, using conditions 1-5, we obtain that $B(\mu) = o[A(\mu)]$ as $\mu \rightarrow +\infty$. To prove this, the integral for $b_n(u, \mu)$ is split into two integrals $b_{n,1}(u, \mu)$ and

$b_{n,2}(u, \mu)$ over the intervals $|x - u| \leq |u|/2 + 1$, $|x - u| \geq |u|/2 + 1$, and then each is estimated separately.

We indicate the method for estimating one term in the expression for $b_{n,1}(u, \mu)$, for example, the integral

$$\begin{aligned} C_{n,j}^2(u, \mu) &= \left| \int_{|x-u| \leq \frac{|u|}{2}+1} \frac{\varphi_n(x)}{\lambda_n + \mu} \{p_j(x) - p_j(u)\} \frac{\partial^{2j}}{\partial x^{2j}} G(\mu; x, u; u) dx \right|^2 \\ &\leq C \frac{(1 + |u|)^{2\alpha_j-2}}{(\lambda_n + \mu)^2} \int_{|x-u| \leq \frac{|u|}{2}+1} \left\{ |x - u| \frac{\partial^{2j}}{\partial x^{2j}} G(\mu; x, u; u) \right\}^2 dx. \end{aligned} \quad (7)$$

From formula (3) it follows directly that

$$\frac{\partial^{2j}}{\partial x^{2j}} G(\mu; x, u; v) = \frac{(-1)^j}{2\pi} \int_{-\infty}^{\infty} \frac{\xi^{2j} e^{i\xi(x-u)}}{l(v, i\xi) + \mu} d\xi.$$

Performing in the right-hand side of the last formula one integration by parts, we find

$$(x - u) \frac{\partial^{2j}}{\partial x^{2j}} G(\mu; x, u; v) = \frac{(-1)^{j+1}}{2\pi i} \int_{-\infty}^{\infty} e^{i\xi(x-u)} \frac{\partial}{\partial \xi} \left[\frac{\xi^{2j}}{l(v, i\xi) + \mu} \right] d\xi.$$

Therefore, from (7) and Parseval's identity it follows that

$$C_{n,j}^2(u, \mu) \leq C \frac{(1 + |u|)^{2\alpha_j-2}}{(\lambda_n + \mu)^2} \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial \xi} \left[\frac{\xi^{2j}}{l(u, i\xi) + \mu} \right] \right|^2 d\xi.$$

Hence, using conditions 1-5, one can show that

$$\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} C_{n,j}^2(u, \mu) du = A(\mu) o(1) \quad \text{as } \mu \rightarrow \infty.$$

The function $b_{n,2}(u, \mu)$ is equal to the sum of many terms. To estimate each term, one must first integrate by parts several times the expression for $G(\mu; x, u; v)$ (depending on the term and on the value of α_j), and then use Parseval's identity and conditions 1-5.

4. Asymptotic formulas for $N(\lambda)$. From the preceding arguments it follows that, under conditions 1-5, the operator $(L + \mu)^{-2k}$ is nuclear and, consequently, the operator L has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots$. Incidentally, we note that if conditions 1-4 are satisfied, but the integral diverges for all $k > 0$, then the

operator L does not have a purely discrete spectrum. Let us find the asymptotics for $N(\lambda)$ as $\lambda \rightarrow +\infty$.

From formula (2) it follows that

$$\int_0^\infty \frac{dN(\lambda)}{(\lambda + \mu)^{2k}} \sim \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{dx d\xi}{\{l(x, i\xi) + \mu\}^{2k}} = \int_0^\infty \frac{d\sigma(\lambda)}{(\lambda + \mu)^{2k}} \quad \text{as } \mu \rightarrow +\infty, \quad (8)$$

where

$$\sigma(\lambda) = \frac{1}{2\pi} \iint_{l(x, i\xi) < \lambda} dx d\xi. \quad (9)$$

Theorem 2. Suppose conditions 1-5 are satisfied. Then

$$N(\lambda) \sim \sigma(\lambda) \quad \text{as } \lambda \rightarrow +\infty. \quad (10)$$

Proof. It is not difficult to verify that the monotone function $\sigma(\lambda)$ satisfies the conditions of B. I. Korenblyum's refinement⁵ of M. V. Keldysh's Tauberian theorem⁴. Therefore (10) follows from (8). Formula (10) is general, but inconvenient for computations. We therefore indicate particular cases in which the asymptotic formula for $N(\lambda)$ can be written more explicitly.

Corollary 1. Suppose conditions 1-5 hold and the limits exist

$$\lim_{|x| \rightarrow \infty} \frac{p_j(x)}{|x|^{\alpha_j}} = A_j, \quad j = 0, 1, \dots, n-1.$$

Suppose there exists a fixed j_0 such that

$$\nu_{j_0} = (n - j_0)/n\alpha_{j_0} \leq (n - j)/n\alpha_j = \nu_j, \quad j = 0, 1, \dots, n-1.$$

Some of the numbers ν_j may be equal to ν_{j_0} . Denote them by $\nu_{j_1}, \dots, \nu_{j_s}$; it is assumed here that the corresponding $A_{j_0}, \dots, A_{j_s}$ are nonzero.

Then

$$N(\lambda) \sim A\lambda^{1/2n + \nu_{j_0}}, \quad (11)$$

where

$$A = \frac{1}{2\pi} \text{mes} \left\{ (x, \xi) : \xi^{2n} + \sum_{k=0}^s A_{j_k} \xi^{2j_k} |x|^{\alpha_{j_k}} \leq 1 \right\}. \quad (12)$$

Remark. From condition 1 it follows that all $p_j(x)$ are positive for large x . This restriction can be removed. In Corollary 1 one may require that $p_{j_0}(x), \dots, p_{j_s}(x)$ satisfy condition 1, while the remaining coefficients instead of this condition satisfy

$$|p_j(x)| \leq C(1 + |x|)^{\alpha_j}.$$

Corollary 2. Suppose

$$l\left(x, \frac{d}{dx}\right)y = y^{(4n)} + (-1)^n(p_n(x)y^{(n)})^{(n)} + p_0(x)y; \quad (13)$$

$p_n(x), p_0(x)$ satisfy conditions 1-5. Then

$$N(\lambda) \sim \frac{1}{2\pi} \int_{p_0(x) \leq \lambda} \left\{ -p_n(x) + [p_n^2(x) - 4(p_0(x) - \lambda)]^{1/2} \right\}^{1/2} dx. \quad (14)$$

In particular, if $p_0(x)$ is bounded, then

$$N(\lambda) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ -p_n(x) + [p_n^2(x) + 4\lambda]^{1/2} \right\}^{1/2} dx. \quad (15)$$

In conclusion it should be noted that the results of the paper generalize to certain elliptic partial differential operators.

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