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Abstract

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MATHEMATICS

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ON PRINCIPAL FIBERINGS WHOSE BASES SATISFY THE SECOND AXIOM OF COUNT- ABILITY

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In the paper of I. M. Gelfand and D. B. Fuks ⁽¹⁾ an extension was considered of the classical category of locally trivial principal G -fiberings, where G is a connected Lie subgroup of the group $GL(n, \mathbf{R})$. The objects of this category were principal G -fiberings with Hausdorff bases, locally trivial in the following sense: for each point x of the base of such a fibering there exists a mapping of the space $E^\#$ of the associated vector fibering into the space $\mathbf{R}^n$, linear on each fiber of the vector fibering and nondegenerate on the fiber over the point x . We emphasize that for fiberings with completely regular bases this notion of local triviality does not differ from the classical one, and that the local triviality of a G -fibering is defined depending not only on the group G , but also on the embedding $G \subset GL(n, \mathbf{R})$. In ⁽¹⁾, in this category, a subcategory was singled out that plays a role analogous to that of the universal G -fibering. It was also indicated there that if the category under consideration is narrowed by requiring the bases of the fiberings to satisfy the second axiom of countability, then there exists a fibering ξ_G such that for any fibering ξ of our category there is a mapping $\xi \rightarrow \xi_G$.

The fibering $(\xi_G)_p : \mathcal{E}_n \rightarrow \mathcal{S}_G$ is constructed as follows. Let $\bar{\mathcal{E}}$ be the product of a countable number of real lines in the Tikhonov topology. By $\mathcal{E}_n$ we denote the set of all nondegenerate n -frames in the linear space $\bar{\mathcal{E}}$. The group $GL(n, \mathbf{R})$ and its subgroup G act on the space $\mathcal{E}_n$ in the obvious way. We denote the quotient space $\mathcal{E}_n/G$ by $\mathcal{S}_G$; the projection $p : \mathcal{E}_n \rightarrow \mathcal{S}_G$ defines, obviously, a principal G -fibering, which is denoted by ξ_G .

All the constructions described carry over without change to the case when G is a Lie subgroup of the group $GL(n, \mathbf{C})$. The corresponding universal fibering will likewise be denoted by $(\xi_G)_p : \mathcal{E}_n \rightarrow \mathcal{S}_G$.

The aim of the present work is to show that the answers to the basic questions

considered in ⁽¹⁾ change essentially under the above narrowing of the category. For example, it turns out that for a wide class of (noncompact) groups G the spaces $\mathcal{S}_G$ are paracompact, from which the traditional consequences are obtained.

Let us list the main results of the paper.

Denote by $\mathfrak{H}$ the class of such subgroups G of the groups $GL(n, \mathbf{C})$ for which the base $\mathcal{S}_G$ of the universal G -fibering is paracompact. Conditions sufficient for G to belong to this class are given by the following theorems:

Theorem 1. *Lie subgroups of the group $L(n, \mathbf{C})$ of matrices of order n whose determinant modulus is equal to one belong to the class $\mathfrak{H}$.*

Theorem 2. *If G is connected, $G \cap L(n, \mathbf{C})$ is a compact group, and in G there exists a matrix C with such eigenvalues λ_1 and λ_2 that $|\lambda_1| > 1$, $|\lambda_2| < 1$, then G belongs to $\mathfrak{H}$.*

The following theorems follow from the definition of the class $\mathfrak{H}$.

Using our narrowed category of G -bundles, we can, analogously to how this was done in ⁽¹⁾, define the groups $H_{\text{ag}}^G(G, \mathbf{R})$.

Theorem 3. If $G \in \mathfrak{S}$, then $H_{\text{alg}}^q(G, \mathbf{R}) = 0$ for $q > 0$.

Theorem 4. Let $G \in \mathfrak{S}$; then for any G -bundle $f: X \rightarrow Y$, the paracompactness of the space X implies the paracompactness of the space Y .

On the other hand, for many noncompact groups the base of the universal bundle has poor topological properties.

Put

$$\|G\| = \inf_{g \in G} \|g\|, \quad \text{where } \|g\| = \sum_{i,j} |x_{ij}| \quad \text{for } g = (x_{ij}).$$

Theorem 5. If $\|G\| = 0$, then every continuous real-valued function on $\mathcal{P}_G$ is constant.

The condition $\|G\| = 0$ is satisfied, for example, for the group $GL(n, \mathbf{C})$. It turns out that for $GL(n, \mathbf{C})$ the groups $H_{\text{alg}}^q(G, \mathbf{R})$ are nontrivial and, moreover, the space $\mathcal{P}_{GL(n, \mathbf{C})}$ can be the base of a nontrivial Hilbert bundle.

Example. Let $H = L_2(\mathbf{C}^n)$ be the space of square-summable functions on $\mathbf{C}^n$. Consider the unitary representation of the group $GL(n, \mathbf{C})$ acting in H , which is given by the formula

$$gf(z) = |\det g| f(g^{-1}z),$$

where $g \in GL(n, \mathbf{C})$, $f(z) \in L_2(\mathbf{C}^n)$. With the help of Theorem 5 one can show that the bundle induced by this representation,

$$\mathcal{E}_n \times H \rightarrow \mathcal{E}_n \times H/G \simeq \mathcal{P}_{GL, \mathbf{C}},$$

with base $\mathcal{P}_{GL}$ and fiber H , is nontrivial.

Analogous results can also be obtained for real linear groups.

Proof of Theorem 1.

Case 1. The group $G = L(n, \mathbf{C})$.

Denote by A the space of all collections $\{x_{p_1, \dots, p_n}\}_{p_i=1,2,\dots}$, where $x_{p_1, \dots, p_n}$ are real numbers, endowed with the Tikhonov topology. The continuous mapping $f^* : \mathcal{E}_n \rightarrow A$, given by the formula

$$f^*(x) = \{(f^*x)_{p_1, \dots, p_n}\},$$

where

$$(f^*x)_{p_1, \dots, p_n} = \left| \det \begin{pmatrix} x_{1p_1} & \dots & x_{1p_n} \\ \dots & \dots & \dots \\ x_{np_1} & \dots & x_{np_n} \end{pmatrix} \right|$$

for $x = \{x_{ij}\}$ (recall that the points of the space $\mathcal{E}_n$ are n -frames in the space $\mathcal{E}$, and each point $x \in \mathcal{E}_n$ can be specified by a collection $\{x_{ij}\}_{i=1, \dots, n; j=1, 2, \dots}$), is constant on the orbits of the group G and therefore defines a continuous mapping $f : \mathcal{P}_G \rightarrow A$. It is obvious that the mapping f takes distinct points to distinct points. We shall show that f is a topological embedding. For this purpose, for every point $y \in f(\mathcal{P}_G) \subset A$ we construct a continuous mapping h of a certain neighborhood of this point (in the space A) into $\mathcal{E}_n$ and show that the mapping $f \circ p \circ h|_{f(\mathcal{P}_G)}$ is the identity. Such a mapping can be defined by setting

$$h(\{y_{p_1, \dots, p_n}\}) = \{y_{ij}\},$$

where

$$y_{ij} = \begin{cases} y_{j, p_1^0, \dots, p_n^0}, & \text{for } i = 1, \\ \frac{y_{p_1^0, \dots, p_{i-1}^0, j, p_{i+1}^0, \dots, p_n^0}}{y_{p_1^0, \dots, p_n^0}}, & \text{for } i > 1 \end{cases}$$

(here $p_1^0, \dots, p_n^0$ is any collection such that

$$y_{p_1^0, \dots, p_n^0} \neq 0$$

).

As is known, A is a metric space. Consequently, $\mathcal{P}_G$ is also a metric space. Since the space $\mathcal{P}_G$ satisfies the second axiom of countability, it is paracompact.

Case 2. The group G is a Lie subgroup of the group $L(n, \mathbf{C})$. In this case there exists a smooth fibration $\mathcal{S}_G \rightarrow \mathcal{S}_{L(n, \mathbf{C})}$ with fiber $L(n, \mathbf{C})/G$. The base of this fibration is paracompact, and the fiber is a finite-dimensional manifold. Consequently, $\mathcal{S}_G$ is also paracompact.

Proof of Theorem 2. Clearly, the group G is not contained in $L(n, \mathbf{C})$. Let $d : G \rightarrow \mathbb{R}$ be the homomorphism defined by the formula

$$d(g) = \ln |\det g|.$$

Denote by Γ the closed normal divisor $G \cap L(n, C)$. By hypothesis Γ is compact. There is an exact sequence

$$0 \rightarrow \Gamma \rightarrow G \rightarrow \mathbb{R} \rightarrow 0.$$

In an obvious way one constructs a homomorphism $\bar{d} : \mathbb{R} \rightarrow G$ such that

$$d\bar{d} : \mathbb{R} \rightarrow \mathbb{R}$$

is the identity mapping. It is clear that $\text{Im } \bar{d}$ is a closed one-parameter subgroup of the group G . Its elements are the matrices $\exp(tA)$, where $t \in \mathbb{R}$, and A is some matrix. We denote this subgroup by B . We may assume that $\exp(A) = C$, where C is the matrix from the hypothesis of the theorem. Then the matrix A , in some basis, has the form:

$$A = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 & 0 \\ 0 & & A' \end{pmatrix}; \quad \exp(tA) = \begin{pmatrix} \exp(t\mu_1) & 0 & \\ 0 & \exp(t\mu_2) & 0 \\ 0 & & \exp(tA') \end{pmatrix},$$

where A' is some matrix and $\text{Re } \mu_1 > 0$, $\text{Re } \mu_2 < 0$. The universal fibration $\mathcal{E}_n \rightarrow \mathcal{S}_G$ decomposes into the composition

$$\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G,$$

where $\mathcal{S}_G$ is obtained after factorizing $\mathcal{S}_\Gamma$ by the action of the group B . We prove that the fibration

$$\mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G$$

is trivial.

For this purpose consider real continuous functions f_1, f_2 on $\mathcal{S}_\Gamma$, defined by the formulas

$$f_1(\Gamma\{x_{ij}\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{\Gamma} |x_{1n}| d\Gamma}{1 + \int_{\Gamma} |x_{1n}| d\Gamma}; \quad f_2(\Gamma\{x_{ij}\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{\Gamma} |x_{2n}| d\Gamma}{1 + \int_{\Gamma} |x_{2n}| d\Gamma},$$

where $\Gamma\{x_{ij}\}$ is the orbit of the element $\{x_{ij}\} \in \mathcal{E}_n$.

On each orbit of the group $B = \{\exp(tA)\}$ the function f_1 increases monotonically, while the function f_2 decreases monotonically (as t increases). Moreover,

$$\lim_{t \rightarrow +\infty} f_1(\exp(tA)x) = a(x) > 0; \quad \lim_{t \rightarrow -\infty} f_2(\exp(tA)x) = b(x) > 0;$$

$$\lim_{t \rightarrow -\infty} f_1 = \lim_{t \rightarrow +\infty} f_2 = 0.$$

The function $f = f_1 - f_2$ on an orbit of the group B increases monotonically from $-b(x)$ to $a(x)$. Therefore the set $f^{-1}(0) \subset \mathcal{S}_\Gamma$ defines a section surface of the fibration

$$\mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G.$$

Consequently, $\mathcal{S}_\Gamma$ is homeomorphic to $\mathcal{S}_G \times \mathbb{R}$. The space $\mathcal{S}_\Gamma$ is paracompact. Consequently, $\mathcal{S}_G$ is also paracompact. Theorem 2 is proved.

Theorem 3 is obvious.

Proof of Theorem 4. According to ⁽¹⁾, there exists a mapping of an arbitrary G -fibration $(\xi)f : X \rightarrow Y$ into the fibration $(\xi_G)p : \mathcal{E}_n \rightarrow \mathcal{S}_G$. Since $\mathcal{S}_G$ is paracompact, there exists a mapping of the fibration (ξ_G) into the classical universal fibration $(\xi_G^*)p_G : EG \rightarrow BG$, described in ⁽²⁾. This latter decomposes into the composition

$$EG \rightarrow B\hat{G} \rightarrow BG,$$

where $\hat{G} \subset G$ is a maximal compact subgroup of the group G . The first arrow denotes a principal $\hat{G}$ -fibration, and the second is a fibration having a section surface. It follows from this that our fibration ξ also decomposes into the composition

$$X \rightarrow \hat{X} \rightarrow Y.$$

Here the first arrow denotes a principal $\hat{G}$ -fibration, and therefore from the paracompactness of X there follows the paracompactness

$\hat{X}$; the second arrow denotes a bundle having a cross-section, and therefore Y is homeomorphic to a subspace of the space $\hat{X}$ and, consequently, is also paracompact.

Proof of Theorem 5. Let h be a nonconstant continuous function on $\mathcal{P}_G$. Denote by f the continuous function on $\mathcal{E}_n$ defined by the formula $f(z) = h(pz)$ for $z \in \mathcal{E}_n$. There exist $x, y \in \mathcal{E}_n$ such that $f(x) \neq f(y)$; put $|f(x) - f(y)| = \varepsilon$. By the definition of the space $\mathcal{E}_n$, its points are n -tuples in $\mathcal{E}$, and each point $z \in \mathcal{E}_n$ can be specified by a set $\{z_{ij} \mid i = 1, \dots, n; j = 1, 2, \dots\}$. Let, for example, $x = \{x_{ij}\}$, $y = \{y_{ij}\}$.

For each natural number k choose a point $z^k = \{z_{ij}^k\} \in \mathcal{E}_n$ such that $z_{ij}^k = 0$ for $j \leq k$. For any element $g \in G$ put $g(z^k) = \{(gz)_{ij}^k\}$. Put $x(g)^k = \{x(g)_{ij}^k\}$, $y(g)^k = \{y(g)_{ij}^k\}$, where

$$x(g)_{ij}^k = \begin{cases} x_{ij}, & (j \leq k), \\ (gz)_{ij}^k, & (j > k); \end{cases} \quad y(g)_{ij}^k = \begin{cases} y_{ij}, & (j \leq k), \\ (gz)_{ij}^k, & (j > k). \end{cases}$$

It is clear that, for sufficiently large k , the sets $x(g)^k$, $y(g)^k$ define elements of the space $\mathcal{E}_n$ and

$$|f(x) - f(x(g)^k)| < \varepsilon/4, \quad |f(y) - f(y(g)^k)| < \varepsilon/4 \quad (1)$$

for every $g \in G$. Since $\|G\| = 0$, one can find such a $g_0 \in G$ that

$$|f(z^k) - f(g_0 x(g_0^{-1})^k)| < \varepsilon/4, \quad |f(z^k) - f(g_0 y(g_0^{-1})^k)| < \varepsilon/4. \quad (2)$$

Since the function f is constant on the orbits of the group G , we have

$$f(g_0 x (g_0^{-1})^k) = f(x (g_0^{-1})^k), \quad f(g_0 y (g_0^{-1})^k) = f(y (g_0^{-1})^k),$$

and from inequalities (1) with $g = g_0^{-1}$ and (2) we obtain $|f(x) - f(y)| < \varepsilon$, which contradicts the definition of the number ε .

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Note: Figure translations are in progress. See original paper for figures.

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