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1969

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Abstract

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UDC 517.964

MATHEMATICS

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ON HÖLDER CONTINUITY OF GENERALIZED SOLUTIONS OF ELLIPTIC EQUATIONS WITH COEFFICIENTS FROM SPACES WITH MIXED NORM

(Presented by Academician S. L. Sobolev on 15 VII 1968)

Consider the equation

$$\frac{\partial}{\partial x_i}(a_{ij}u_{x_j} + a_i u + f_i) + b_i u_{x_i} + a u = f \quad (1)$$

under the condition that the leading coefficients $a_{ij}(x)$ are bounded and the equation is strictly elliptic, i.e., under the condition

$$\nu \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq \mu \xi_i \xi_i; \quad \nu, \mu = \text{const} > 0. \quad (2)$$

With respect to the lower-order coefficients and the free terms we shall assume

$$\|a_i^2; b_i^2; f_i^2; a, f\|_{L_{(r_1, r_2)}(D)} \leq \mu; \quad (3)$$

$$(r_1, r_2) \in \Omega_2^{(s)}. \quad (4)$$

(For the definitions of the space $L_{(r_1, r_2)}(D)$ and the set $\Omega_2^{(s)}$, see (8).)

Let us note that, when condition (4) is satisfied, the spaces $L_{(r_1, r_2)}(D)$ contain elements $f(x)$ belonging to $L_{1+\alpha}$ (α is a sufficiently small positive number), but not belonging to the space $L_{n/2}(D)$.

In the present work it is proved that every generalized solution $u(x)$ from $W_2^1(D)$ of equation (1), when conditions (2), (3), (4) are fulfilled, belongs to the class $C_{0,\alpha}$.

In the particular case $r_1 = r_2 = q/2$, $q > n$, i.e., when

$$\|a_i^2; b_i^2; f_i^2; a, f\|_{L_{q/2}(D)} \leq \mu, \quad (5)$$

the membership of a generalized solution $u(x)$ from $W_2^1(D)$ in $C_{0,\alpha}$ had previously been established in a number of works ⁽³⁻⁵⁾. We note that the study of the smoothness of a generalized solution in the case $r_1 = r_2 = q/2$, $q > n$, and in the case considered by us, differ substantially.

The works mentioned above in one way or another rely on Sobolev imbedding theorems ⁽¹⁾, which are not applicable in our case. This is also understandable, since the spaces $L_{(r_1, r_2)}(D)$ contain functions belonging to L_p , $1 < p \leq n/2$.

Thanks to the introduction of spaces with mixed norm $L_{(r_1, r_2)}(D)$ and the establishment for them of imbedding theorems and certain integral estimates ⁽⁹⁾, and also thanks to a generalization of De Giorgi's lemma and of a number of results of O. A. Ladyzhenskaya and N. N. Ural'tseva, it has been possible to study membership in the Hölder class of generalized solutions of a broad class of elliptic equations of the form (1).

We shall state the results obtained. First let us introduce some notation. Let D be an n -dimensional bounded domain belonging to E^n ; let s be a natural number less than n ; let K_ρ be a ball in E^n of radius ρ ; $D_\rho = \overline{K}_\rho \cap D$; $\text{osc}\{u(x); D\}$ the oscillation of $u(x)$ on D , i.e., the difference between $\text{vrai max}_D u(x)$

and $\text{vrai min}_D u(x)$; $A_{k,\rho}$ is the set of points x in K_ρ for which $u(x) > k$; $u^{(k)}(x) = \max\{u(x) - k; 0\}$.

Definition. The boundary S of the domain D (or its part S_1) is said to satisfy condition (A) if there exist two positive numbers a_0 and θ_0 such that, for every ball K_ρ with center on S (respectively on S_1) of radius $\rho \leq a_0$ and for any component $\widetilde{D}_\rho$ of the intersection D_ρ of the ball K_ρ with D , the equality

$$\text{mes } \widetilde{D}_\rho \leq (1 - \theta_0) \text{mes } K_\rho.$$

Lemma 1. If $s/q_1 + (n-s)/q_2 > n-1$, $u(x) \in W_1^1(K_\rho)$, Ω is a measurable set belonging to K_ρ , $\Omega_0 = \{x \in K_\rho; u(x) \equiv 0\}$, then

$$\|u\|_{L_{(q_1, q_2)}(\Omega)} \leq \beta \frac{\rho^{s/q_1 + (n-s)/q_2 + 1}}{\text{mes } \Omega_0} \int_{K_\rho} |\text{grad } u| dx;$$

β is a constant depending on n, s .

Corollary 1. If $s/q_1 + (n-s)/q_2 > n-1$, then for any function $u(x)$ from $W_1^1(K_\rho)$, for arbitrary k and l ($l \geq k$), the estimate holds

$$(l-k)\|1\|_{L(q_1, q_2)(A_{l, \rho})} \leq \beta \frac{\rho^{s/q_1 + (n-s)/q_2 + 1}}{\text{mes}(K_\rho \setminus A_{k, \rho})} \int_{A_{k, \rho} \setminus A_{l, \rho}} |\text{grad } u| dx,$$

where β is the constant from Lemma 1.

Remark. For $q_1 = q_2 = 1$ we obtain the well-known result of De Giorgi ⁽⁶⁾.

Let M, γ, δ be fixed numbers; let the numbers p_1, p_2 and $\varepsilon > 0$ satisfy the condition

$$s/p_1 + (n-s)/p_2 = n/2 - 1 + \varepsilon.$$

Definition. Denote by $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ the set of functions $u(x)$ from $W_2^1(D)$ with $\text{vrai max}_D |u| \leq M$ and such that, for $\pm u(x)$ in any ball $K_\rho \subset D$ and for any $\sigma \in (0, 1)$, the inequalities

$$\int_{A_{k, \rho(1-\sigma)}} |\text{grad } u(x)| dx \leq \gamma \left[(\sigma\rho)^{-2} \int_{A_{k, \rho}} |u^{(k)}|^2 dx + \|1\|_{L(p_1, p_2)(A_{k, \rho})}^2 \right] \quad (6)$$

hold for $k \geq \text{vrai max}_{K_\rho} u(x) - \delta$.

Lemma 2. Let $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ and $K_{\rho_0} \subset D$, where

$$\rho_0^\varepsilon < \text{vrai max}_{K_{\rho_0}} u(x) - k \equiv H < \delta;$$

then there exists a number θ such that from

$$\|1\|_{L(p_1, p_2)(A_{k, \rho_0})} \leq \theta \rho_0^{n/2 - 1 + \varepsilon}$$

it follows that

$$\text{vrai max}_{K_{\rho_0/2}} u(x) \leq \frac{1}{2} (\text{vrai max}_{K_{\rho_0}} u(x) + k).$$

Lemma 3. There exists a positive number σ such that, for any ball K_ρ ($\rho < 1$) belonging to D together with the concentric ball $K_{4\rho}$, and for any function $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$, at least one of the two inequalities holds:

$$\text{osc}\{u, K_\rho\} \leq 2^\sigma \rho^\varepsilon$$

or

$$\text{osc}\{u, K_\rho\} \leq (1 - 1/2^{\sigma-1}) \text{osc}\{u, K_{4\rho}\}.$$

From Lemma 3 and Lemma 4.8 ⁽²⁾, p. 90, it follows

Theorem 1. Let $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ and $K_{\rho_0} \subset D$ ($\rho_0 \leq 1$); then, for any ball K_ρ , $\rho \leq \rho_0$, concentric with K_{ρ_0} , the estimate holds

$$\text{osc}\{u, K_\rho\} \leq c(\rho/\rho_0)^\alpha,$$

where $a = \min\{-\lg_4(1 - 1/2^{\sigma-1}), \varepsilon\}$, $c = 4^a \max\{2M; 2^\sigma \rho_0^e\}$. The number σ is taken from Lemma 3.

To study the smoothness of functions from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ in domains adjacent to the boundary, we single out from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ a narrower class of functions satisfying inequalities of type (6) not only for interior balls, but also for balls intersecting the boundary S of the domain.

Definition. Denote by $B(D \cup S_1, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ the set of functions $u(x)$ from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ which, together with $u(x)$, satisfy inequalities (6) also for balls K_ρ intersecting S_1 , but not intersecting $S \setminus S_1$, for $k \geq \text{vrai max}_{K_\rho \cap D} u(x) - \delta$ and $k \geq \text{vrai max}_{K_\rho \cap S_1} u(x)$.

The domains of integration $A_{k,\rho}$ for balls intersecting S_1 in (6) are defined as the sets of points of $K_\rho \cap D$ at which $u(x) > k$.

Lemma 4. There exists a positive number σ such that for any ball K_ρ and the concentric ball $K_{4\rho}$ with center on S and $4\rho \leq a/4$ ($a = \min(a_0; 1)$), and any function $u(x)$ from $B(\overline{D}, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ satisfying the condition

$$\text{osc}\{u; S \cap K_\rho\} \leq b\rho^e, \quad e > 0,$$

at least one of the two inequalities holds:

$$\text{osc}\{u; K_\rho \cap D\} \leq 2^\sigma \rho^{\varepsilon_1}, \quad \varepsilon_1 = \min\{\varepsilon, e\},$$

or

$$\text{osc}\{u; K_\rho \cap D\} \leq (1 - 1/2^{\sigma-1}) \text{osc}\{u, K_{4\rho} \cap D\}.$$

Theorem 2. If S satisfies condition (A) and the function $u(x)$ from $B(D \cup S, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ satisfies on S the condition

$$\text{osc}\{u; S \cap K_\rho\} \leq b\rho^e, \quad e > 0$$

for balls with centers on S and radius $\rho \leq a_0$, then $u(x)$ satisfies a Hölder condition in $\overline{D}$, i.e. for any ball K_ρ

$$\text{osc}\{u, K_\rho \cap D\} \leq c\rho^\alpha.$$

The constants c and α are determined by known parameters.

On the basis of the results set forth and the results of work ², the validity of the following theorem is established.

Theorem 3. Under conditions (2), (3), (4), the generalized solution $u(x)$ from $W_2^1(D)$ of equation (1) belongs to the class $C_{0,\alpha_1}(\overline{D'})$, where D' is an interior subdomain of the domain D .

If, moreover, $\text{vrai max}_S u(x) < M$, then $u(x) \in C_{0,\alpha_2}(D)$.

If the boundary S satisfies condition (A) and $u|_S \in C_{0,\beta}(S)$, then $u(x) \in C_{0,\alpha_3}(\overline{D})$.

In all cases α_i and $|u|_{\alpha_i,D}$ are determined and estimated, respectively, by known quantities.

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Received

8 VII 1968

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