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1969

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Abstract

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UDC 513.83

MATHEMATICS

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A TOPOLOGICAL VARIANT OF SŁUPECKI'S THEOREM FOR SOME COMPACTA

(Presented by Academician P. S. Aleksandrov on 31 I 1969)

In mathematical logic (algebras of logic) the following theorem of Słupecki is known.

Theorem 1. *Let X be an arbitrary set. There exists a function $\varphi : X^2 \rightarrow X$ of two variables on X with values in X such that, for any integer $n \geq 1$, every function $f : X^n \rightarrow X$ is representable as a superposition of the function φ and functions of one variable $X \rightarrow X$.*

The aim of the present note is to extend Słupecki's theorem (restricting ourselves, naturally, to the consideration of continuous functions) to certain topological spaces, such as the Cantor discontinuum, the Hilbert cube, or the torus. The possibility of such an extension follows in almost all cases from a certain analogue of the Kolmogorov–Arnold theorems ^(1, 2), which solve Hilbert's famous 13th problem; we give the formulation of the theorem in the modification due to Ostrand ⁽³⁾.

Theorem 2. *Let X_p , $p = 1, 2, \dots, n$, be finite-dimensional compacta,*

$$\dim X_p = d_p, \quad \sum_{p=1}^n d_p = m.$$

There exist continuous functions $\psi^{pq} : X_p \rightarrow [0, 1]$, $p = 1, 2, \dots, n$; $q = 1, 2, \dots, 2m + 1$, such that every continuous mapping $f : \prod_{p=1}^n X_p \rightarrow R$ admits a representation

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2m+1} \eta^q \left(\sum_{p=1}^n \psi^{pq}(x_p) \right), \quad (1)$$

where $\eta^q : R \rightarrow R$ are certain continuous functions (continuous coordinates of f with respect to the system ψ^{pq}).

We shall prove a theorem analogous to Theorem 2 for mappings into the circle, more precisely, Kolmogorov's theorem for mappings of a polyhedron into the circle.

Theorem 3. *Let X be a connected finite polyhedron of dimension d , let $n \geq 1$ be any integer and $m = nd$. There exist systems of continuous mappings $\psi^{pq} : X \rightarrow S$, $p = 1, \dots, n$, $q = 1, \dots, 2m + 1$, and $\psi^\alpha : X \rightarrow S$, $\alpha \in H^1(X)$, such that every continuous mapping $f : X^n \rightarrow S$ of the n -th power of X into the circle S , regarded with its additively written natural group structure, admits the representation*

$$f(x_1, \dots, x_n) = \sum_{p=1}^n \psi_p^\alpha(x_p) + \sum_{q=1}^{2m+1} \eta^q \left(\sum_{p=1}^n \psi^{pq}(x_p) \right), \quad (2)$$

where $\eta^q : S \rightarrow S$ are certain continuous mappings, and $H^1(X)$ is the one-dimensional integral cohomology group of the polyhedron X .

We first construct the mappings $\psi^{pq} : X \rightarrow S$ and verify that every mapping $f : X^n \rightarrow S$ homotopic to zero admits a decomposition in terms of the mappings ψ^{pq} . Namely, set $\psi^{pq} = i\bar{\psi}^{pq}$, where $\bar{\psi}^{pq} : X \rightarrow [0, 1]$, $p =$

$= 1, \dots, n$; $q = 1, \dots, 2m + 1$, constructed on the basis of Theorem 2 for the compactum X and the number n , and we shall assume that $\bar{\psi}^{pq}(X) \subset [0, 1/4\pi]$; and let $i : [0, 1/2] \rightarrow S$ be the restriction, naturally nonessential, while $i : [0, 1/2] \rightarrow S$ is the standard embedding. Using the homotopy of f to zero, by the well-known property of a covering one can find a continuous mapping $\tilde{f} : X \rightarrow R$ such that $f = g\tilde{f}$, where $g : R/N \rightarrow S$ is the standard homeomorphism of the covering of the circle by the line. By Theorem 2 we find such $\tilde{\eta}^q : R \rightarrow R$ that

$$\tilde{f}(x_1, \dots, x_n) = \sum_{q=1}^{2m+1} \tilde{\eta}^q \left(\sum_{p=1}^n \tilde{\psi}^{pq}(x_p) \right), \quad (3)$$

and, in view of the restrictions on ψ^{pq} , we may assume that the mappings $\tilde{\eta}^q$ are defined only on the segment $[0, 1/2]$, and therefore they can be extended, without changing formula (3), to the segment $[0, 1]$, taking $\tilde{\eta}^q(0) = \tilde{\eta}^q(1)$. Applying to both sides of equality (3) the homeomorphism g , we obtain a decomposition of the form (1), where $\eta^q = g\tilde{\eta}^q : S \rightarrow S$ are certain continuous mappings. Taking, further, $\psi^0 : X \rightarrow 0 \in S$ and $\alpha_p = 0$, $p = 1, \dots, n$, we obtain the required (2).

We now construct the remaining mappings of the system $\psi^\alpha : X \rightarrow S$. As is known, the set of homotopy classes of mappings of any topological space Y into the circle S , with respect to the operation induced by the operation in S , forms an abelian group (the Brouwer group $\pi^1(Y)$), in the polyhedral case isomorphic to the one-dimensional integral cohomology group $H^1(Y)$ of the space Y . Let $\xi : H^1(X) \approx \pi^1(X)$ be the corresponding natural isomorphism,

and let $\psi^\alpha : X \rightarrow S$ be any representative of the class $\xi\alpha$, $\alpha \in H^1(X)$, where as $\psi^0 : X \rightarrow S$ we take, as was mentioned above, the mapping $\psi^0 : X \rightarrow 0 \in S$.

Now let an arbitrary continuous mapping $f : X^n \rightarrow S$ be given. Using the trivialization in dimension 1 of Künneth's formula, we have the decomposition

$$\pi^1(X^n) \approx \pi^1(X) \oplus \dots \oplus \pi^1(X),$$

and, by virtue of the naturality of the isomorphism ξ and of the isomorphism participating in Künneth's formula, we conclude that f is homotopic to the sum $\sum_{p=1}^n f_p$ of continuous mappings, each of which depends only on one variable:

$$f_p(x_1, \dots, x_n) = f_p(x_0, \dots, x_0, x_p, x_0, \dots, x_0),$$

and therefore one can find such $\alpha_p \in H^1(X)$, $p = 1, \dots, n$, that $f - \sum_{p=1}^n \psi^{\alpha_p} \sim 0$, and, by what was proved above,

$$f(x_1, \dots, x_n) - \sum_{p=1}^n \psi^{\alpha_p}(x_p) = \sum_{q=1}^{2m+1} \eta^q \left(\sum_{p=1}^n \psi^{p,q}(x_p) \right)$$

for some continuous $\eta^q : S \rightarrow S$, whence (2) follows.

Corollary (Kolmogorov's theorem for mappings of a polyhedron into a torus). *Let X be a connected finite polyhedron of dimension d , and let $T^k = \prod_{i=1}^k Y_i$, $Y_i = S$, be the k -dimensional torus. For any integer $n \geq 1$ one can specify mappings $\psi^\alpha : X \rightarrow T^k$ and $\psi^{p,q} : X \rightarrow T^k$, $\alpha \in H^1(T^k)$, $p = 1, \dots, n$, $q = 1, \dots, 2m+1$, such that every continuous mapping $f : X^n \rightarrow T^k$ is representable in the form (2), where $\eta^q : T^k \rightarrow T^k$ are certain continuous mappings, and the sum is the natural group operation in the direct sum of k copies of the group S .*

Indeed, let $n \geq 1$ be given. On the basis of Theorem 3 we construct systems $\psi^{p,q} : X \rightarrow S$, $p = 1, \dots, n$; $q = 1, \dots, 2m+1$; $\psi^\alpha : X \rightarrow S$, $\alpha \in H^1(X)$. Let, further, $f : X^n \rightarrow T^k$ be any continuous mapping. Denote by $p_j : T^k \rightarrow S$, $j = 1, \dots, k$, the natural projections of the torus onto its factors and consider—

consider the continuous mappings $f_j : X^n \rightarrow S$, given by the equalities $f_j = p_j f$. By Theorem 3 there exist continuous $\eta_j^q : S \rightarrow S$ and $\alpha_{pj} \in H^1(X)$, $j = 1, \dots, k$, such that

$$f_j(x_1, \dots, x_n) = \sum_{p=1}^n \tilde{\psi}^{\alpha_{pj}}(x_p) + \sum_{q=1}^{2m+1} \eta_j^q \left(\sum_{p=1}^n \tilde{\psi}^{p,q}(x_p) \right).$$

But then, introducing the mappings $\eta^q(t_1, \dots, t_k) = (\eta_1^q(t_1), \dots, \eta_k^q(t_k))$, $\psi^{p,q}(x) = (\tilde{\psi}^{p,q}(x), \dots, \tilde{\psi}^{p,q}(x))$; $\psi^{\alpha_p}(x) = (\tilde{\psi}^{\alpha_{p1}}(x), \dots, \tilde{\psi}^{\alpha_{pk}}(x))$, and observing, by virtue

of the decomposition $H^1(T^k) \approx H^1(S) \oplus \dots \oplus H^1(S)$, that the collection of mappings ψ^α may be regarded as indexed precisely by the group $H^1(T^k)$, we obtain the required (2).

Let us note that the consequence just proved could also have been stated separately, proving Theorem 3 precisely in this more general formulation; this was not done only in order to simplify the notation.

We pass to the “continuous” Ślupecki theorem.

Theorem 4. *Let X be one of the following compacta: the Cantor discontinuum D^τ ; a k -dimensional torus; a k -dimensional cube; a generalized cylinder (the product of a k -dimensional torus and an l -dimensional cube). There exists a continuous mapping $\varphi : X^2 \rightarrow X$ such that, for every integer $n \geq 1$, every continuous mapping $f : X^n \rightarrow X$ is representable as a superposition of the mapping φ and continuous mappings of X into X .*

1. $X = D^\tau = \prod_{\alpha \in T} A_\alpha$; $A_\alpha = \{0, 1\}$. In this case the space X admits the structure of an Abelian group, and as the operation $\varphi : X^2 \rightarrow X$ we shall take precisely the group operation. Since the set of indices T may be regarded as infinite (otherwise X is finite and discrete, and Theorem 4 coincides with Theorem 1), there exists a one-to-one mapping $\xi : T \rightarrow T \times \{1, \dots, n\}$ which is “onto.” It generates a homeomorphism $\psi : (D^\tau)^n \rightarrow D^\tau$, defined by the formula

$$\{\psi(x_1, \dots, x_n)\}_\alpha \in \{x_k\}_\beta,$$

where $\xi(\alpha) = (\beta, k)$, and by $\{x\}_\alpha$ is denoted the α -th coordinate of the point $x \in D^\tau$. Obviously, for every continuous $f : X^n \rightarrow X$ there exists a continuous $\eta : X \rightarrow X$ such that $f = \eta\psi$,—it suffices to put $\eta = f\psi^{-1}$. It remains to show that the mapping ψ itself is the sum (with respect to the group operation in X) of some n fixed (not depending on f) mappings ψ^q . But such mappings are constructed trivially:

$$\psi^q(x) = \psi(0, \dots, 0, x, 0, \dots, 0), \quad x \in D^\tau = X,$$

where $0 \in X$ is the point with zero coordinates, and x stands in $\psi(0, \dots, x, \dots, 0)$ in the k -th place. Then

$$\begin{aligned} \psi(x_1, \dots, x_n) &= \sum_{p=1}^n \psi^p(x_p), \\ f(x_1, \dots, x_n) &= \eta \left(\sum_{p=1}^n \psi^p(x_p) \right). \end{aligned}$$

2. $X = S$ or $X = T^k$. In these cases Theorem 4 is a trivial consequence of Theorem 3 or of its corollary. Note that in these cases the formula (2) itself can be somewhat simplified by reducing to a finite number the number of mappings of the group $\psi^\alpha : X \rightarrow X$. Thus, for $X = S$ and any continuous $f : X^n \rightarrow X$ the decomposition

$$f(x_1, \dots, x_n) = \sum_{p=1}^n \tilde{\eta}^q(\psi(x_p)) + \sum_{q=1}^{2m+1} \eta^q \left(\sum_{p=1}^n \psi^{pq}(x_p) \right),$$

holds, where $\psi(x) = x$ for all $x \in X = S$, and the ψ^{pq} are the same as in Theorem 3.

3. $X = I$ or $X = I^k$. Słupecki's theorem in this case is not difficult to derive from Kolmogorov's theorem. The only complication is that, with respect to addition, I is not a group, and therefore another operation on X is required. However, the required operation is easily constructed: $\varphi(x, y) = \min(1, X + Y)$.

4. $X = T^k \times I^l = \prod_{i=1}^{k+l} Y_i$, $Y_i = S$, $i \leq k$, $Y_i = I$, $i > k$.

Construct the systems

$$\psi_1^{pq} : X \rightarrow T^k; \quad \psi_1^\alpha : X \rightarrow T^k; \quad \psi_2^{pq} : X \rightarrow I^l,$$

$$p = 1, \dots, n, \quad q = 1, \dots, 2n+1; \quad \alpha \in H^1(x)$$

in accordance with Theorems 3 and 4 for mappings of X , respectively, into T^k and I^l . For any continuous mapping $f : X^n \rightarrow X$, the mappings generated by it, $p_1 f : X \rightarrow T^k$ and $p_2 f : X \rightarrow I^l$, where $p_1 : T^k \times I^l \rightarrow T^k$ and $p_2 : T^k \times I^l \rightarrow I^l$ are projections, are representable, by the corresponding theorems, in the form

$$f_i(x_1, \dots, x_n) = \sum_{p=1}^n \psi_i^{\alpha p}(x_p) - \eta_i \left(\sum_{q=1}^{2m+1} \eta_i^q \left(\sum_{p=1}^n \psi_1^{pq}(x_p) \right) \right), \quad i = 1, 2,$$

where, for symmetry of the operation φ , in all cases the summation sign denotes $\psi_2^{\alpha p} = 0$ and $\eta_1(t) = t$. But then

$$f(x_1, \dots, x_n) = \sum_{p=1}^n \psi^{\alpha p}(x_p) + \eta \left(\sum_{q=1}^{2m+1} \eta^q \left(\sum_{p=1}^n \psi^{pq}(x_p) \right) \right),$$

where $\psi^{\alpha p}(x) = (\psi_1^{\alpha p}(x), \psi_2^{\alpha p}(x))$; $\eta(t_1, t_2) = (\eta_1(t_1), \eta_2(t_2))$; $\psi^{pq}(x) = (\psi_1^{pq}(x), \psi_2^{pq}(x))$, and the summation operation $X^2 \rightarrow X$ is coordinatewise (the group operation in the torus and that generated by φ in the cube).

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Received
30 XII 1969

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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