

SOME ESTIMATES FOR A SINGULAR INTEGRAL WITH SUMMABLE DENSITY

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Abstract

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MATHEMATICS

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SOME ESTIMATES FOR A SINGULAR INTEGRAL WITH SUMMABLE DENSITY

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Let the function $u(x)$ be summable on (a, b) and belong to L_p ($p > 1$) on every segment $[a + \xi, b - \eta]$ ($\xi, \eta > 0$). Introduce the functions

$$\Omega_p(u, \xi, \eta) = \left\{ \int_{a+\xi}^{b-\eta} |u(x)|^p dx \right\}^{1/p},$$

$$\omega_p(u, \tau, \xi, \eta) := \sup_{h \in A} \left\{ \int_{a+\xi}^{b-\eta-h} |u(x+h) - u(x)|^p dx \right\}^{1/p},$$

where $\xi, \eta, \tau > 0$; $\xi + \eta \leq b - a = l$; $1 < p \leq \infty$; $A = \{h; 0 < h \leq \min\{\tau, l - \xi - \eta\}\}$.

For $p = +\infty$

$$\Omega_p(u, \xi, \eta) = \max_{x \in [a+\xi, b-\eta]} |u(x)| = \Omega(u, \xi, \eta),$$

$$\omega_p(u, \tau, \xi, \eta) = \max_{\substack{x, y \in [a+\xi, b-\eta] \\ |x-y| \leq \tau}} |u(x) - u(y)| = \omega(u, \tau, \xi, \eta).$$

Denote

$$\tilde{u}(x) = \int_a^b \frac{u(s)}{s-x} ds = \lim_{\varepsilon \rightarrow +0} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{u(s)}{s-x} ds.$$

In the present work we consider the question of the relation between the ordered pairs

$(\Omega_p(\tilde{u}, \xi, \eta), \omega_p(\tilde{u}, \tau, \xi, \eta))$ and $(\Omega_p(u, \xi, \eta), \omega_p(u, \tau, \xi, \eta))$.

This problem in the case $p = +\infty$ (in the class of functions continuous on (a, b)) was posed in ⁽¹⁾ and solved in the works ^(1,2).

Theorem 1. Let $1 < p \leq \infty$. If the integrals

$$\int_0^{\xi} \frac{\Omega_p(u, t, t)}{t^{1/p}} dt, \quad \int_0^{\eta/2} \frac{\omega_p(u, t, \xi/2, \eta/2)}{t} dt,$$

converge, then for $0 < \xi, \eta \leq b/2$, $\delta > 0$, the estimates

$$\begin{aligned} \Omega_p(\tilde{u}, \xi, \eta) \leq cq \left\{ \int_0^{l/2} \frac{\Omega_p(u, t, l/4)}{t^{1/p}(t + \xi)^{1/q}} dt + \int_0^{l/2} \frac{\Omega_p(u, l/4, t)}{t^{1/p}(t + \eta)^{1/q}} dt + \right. \\ \left. + \int_0^{\xi/2} \frac{\omega_p(u, t, \xi/2, l/4)}{t} dt + \int_0^{\eta/2} \frac{\omega_p(u, t, l/4, \eta/2)}{t} dt \right\}, \end{aligned} \quad (1)$$

$$\begin{aligned} \omega_p(\tilde{u}, \delta, \xi, \eta) \leq cq \left\{ \frac{\delta}{\xi + \delta} \int_0^{l/2} \frac{\Omega_p(u, t, l/4)}{t^{1/p}(t + \xi)^{1/q}} dt + \frac{\delta}{\eta + \delta} \int_0^{l/2} \frac{\Omega_p(u, l/4, t)}{t^{1/p}(t + \eta)^{1/q}} dt + \right. \\ \left. + \delta \int_0^{\xi/2} \frac{\omega_p(u, t, \xi/2, l/4)}{t(t + \delta)} dt + \delta \int_0^{\eta/2} \frac{\omega_p(u, t, l/4, \eta/2)}{t(t + \delta)} dt \right\}, \end{aligned} \quad (2)$$

where $1/p + 1/q = 1$; c is a constant depending only on l .

Note that for $p = +\infty$ these estimates turn into the estimates of the work ⁽²⁾, which is a refinement and development of the work ⁽¹⁾.

Let us consider some constructions based on the preceding estimates. Denote by G ⁽²⁾ the set of ordered pairs of functions $(\varphi(\xi, \eta), \psi(\delta, \xi, \eta))$, defined respectively on $\{0 < \xi, \eta \mid \xi + \eta \leq l\}$, $\{0 < \delta, \xi, \eta \mid \delta + \xi + \eta \leq l\}$, and satisfying the conditions:

- 1) $\varphi(\xi, \eta)$, $\psi(\delta, \xi, \eta)/\delta$ are positive and almost decreasing* in each of the arguments uniformly with respect to the others;
- 2) $\lim_{\delta \rightarrow +0} \psi(\delta, \xi, \eta) = 0$.

By definition, a function $u(x)$, given on (a, b) , belongs to the set $H_{\varphi\psi}^p$ if there exist constants $c_1(u), c_2(u) > 0$ such that

$$\Omega_p(u, \xi, \eta) \leq c_1(u)\varphi(\xi, \eta), \quad \omega_p(u, \delta, \xi, \eta) \leq c_2(u)\psi(\delta, \xi, \eta),$$

where $(\varphi, \psi) \in G$.

By introducing the norm

$$\|u\|_{\varphi\psi}^p = \max \left\{ \sup_{\xi, \eta} \frac{\Omega_p(u, \xi, \eta)}{\varphi(\xi, \eta)}, \sup_{\xi, \eta, \delta} \frac{\omega_p(u, \delta, \xi, \eta)}{\psi(\delta, \xi, \eta)} \right\},$$

$H_{\varphi\psi}^p$ is turned into an infinite-dimensional Banach space.

Theorem 2. Let $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in G$.

Then:

- a) if $\varphi_1 \sim \varphi_2, \psi_1 \sim \psi_2$, then $H_{\varphi_1\psi_1}^p$ and $H_{\varphi_2\psi_2}^p$ coincide**;
- b) if the limiting relations

$$\lim_{\delta \rightarrow +0} \frac{\psi_1(\delta, \xi, \eta)}{\psi_2(\delta, \xi, \eta)} = 0, \quad \lim_{\xi \rightarrow +0} \frac{\psi_1(\delta, \xi, \eta)}{\psi_2(\delta, \xi, \eta)} = 0,$$

$$\lim_{\eta \rightarrow +0} \frac{\psi_1(\delta, \xi, \eta)}{\psi_2(\delta, \xi, \eta)} = 0, \quad \lim_{\xi \rightarrow +0} \frac{\varphi_1(\xi, \eta)}{\varphi_2(\xi, \eta)} = 0, \quad \lim_{\eta \rightarrow +0} \frac{\varphi_1(\xi, \eta)}{\varphi_2(\xi, \eta)} = 0,$$

are satisfied uniformly, then $H_{\varphi_1\psi_1}^p$ is a proper part of $H_{\varphi_2\psi_2}^p$, and the embedding is completely continuous.

Denote by Φ the set of ordered pairs of functions $(\varphi, \psi) \in G$ satisfying the conditions:

- 1) $\psi(\delta, \xi, \eta)$ almost increases with respect to δ ;
- 2) $\psi(\delta_1 + \delta_2, \xi, \eta) = O(\psi(\delta_1, \xi, \eta) + \psi(\delta_2, \xi, \eta))^{***}$;
- 3) $\psi(\delta, \xi, \eta) = O(\varphi(\xi, \eta))$.

Following (2), introduce the set H_p of ordered pairs of functions $(\varphi(\xi), \psi(\delta, \xi))$ satisfying the conditions:

- 1) $\varphi(\xi) > 0, \psi(\delta, \xi) > 0$;
- 2)

$$\int_0^{1/2} \frac{\varphi(t)}{t^{1/p}(t + \xi)^{1/q}} dt = O(\varphi(\xi));$$

3)

$$\delta \int_0^\xi \frac{\psi(t, \xi/2)}{t(t + \delta)} dt = O(\psi(\delta, \xi));$$

4)

$$\frac{\delta}{\xi + \delta} \varphi(\xi) = O(\psi(\delta, \xi)).$$

By definition $(\varphi, \psi) \in \Phi H_p$ if $(\varphi, \psi) \in \Phi$ and $(\varphi(\xi, l/4), \psi(\delta, \xi, l/4)), (\varphi(l/4, \eta), \psi(\delta, l/4, \eta)) \in H_p$.

Theorem 3. Let $(\varphi, \psi) \in \Phi H_p$. Then the operator

$$Au = \int_a^b \frac{u(s)}{s - x} ds$$

acts in $H_{\varphi\psi}^p$ and is bounded.

* A nonnegative function $f(x)$, defined on a set $\chi \subset (-\infty, +\infty)$, is called almost increasing (almost decreasing) if there exists a constant $c > 0$ such that the inequality $x_1 \leq x_2$, $x_1, x_2 \in \chi$, implies the inequality $f(x_1) \leq cf(x_2)$ ($f(x_1) \geq cf(x_2)$).

** Nonnegative functions $f(x)$ and $g(x)$, defined on χ , are called equivalent ($f \sim g$) if there exist constants $B_1, B_2 > 0$ such that for every $x \in \chi$ the inequalities $B_1 f(x) \leq g(x) \leq B_2 f(x)$ hold.

*** Here and in what follows, uniform satisfaction of the O -relation is assumed.

This theorem, in the case $p = +\infty$, was proved in ².

It is easy to verify that the pair of functions

$$\varphi(\xi, \eta) = \frac{1}{\xi^\alpha} + \frac{1}{\eta^\beta}, \quad \psi(\delta, \xi, \eta) = \frac{\delta^\alpha}{\xi^\alpha(\xi + \delta)^\alpha} + \delta^\gamma + \frac{\delta^\beta}{\eta^\beta(\eta + \delta)^\beta}$$

$$(0 < \alpha, \beta < 1/q; 0 < \gamma < 1)$$

belongs to ΦH_p .

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Note: Figure translations are in progress. See original paper for figures.

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