

ON THE STABILIZATION OF THE SOLUTION OF THE HEAT-CONDUCTION EQUATION

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Abstract

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MATHEMATICS

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ON THE STABILIZATION OF THE SOLUTION OF THE HEAT-CONDUCTION EQUATION

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Consider the equation

$$\partial u(x, t) / \partial t = \Delta u(x, t), \quad x = [x_1, x_2, \dots, x_n] \in R^n, \quad (1)$$

with the initial condition

$$u(x, t)|_{t=0} = u_0(x). \quad (2)$$

In [4] it was proved that, in the class of bounded continuous initial functions, a sufficient condition for stabilization of the solution of the Cauchy problem (1)–(2) is the existence of a spherical limiting mean of $u_0(x)$. In [5] it was proved that this condition is also necessary. In the present paper the generalized Cauchy problem (1)–(2) is considered in the class of initial functionals $u_0(x) \in S'_{1/2}(R^n)$, and the behavior of the solution as $t \rightarrow +\infty$ is studied. At the beginning of the article a generalized spherical limiting mean of the functional $u_0(x)$ is defined, and it is shown that if it exists, then the solution stabilizes to it. Next it is proved that the generalized spherical limiting mean is a harmonic function. In the class of positive initial functionals $u_0(x)$, the existence of a generalized spherical limiting mean is not only a sufficient but also a necessary condition for stabilization of the solution. In this class the solution can stabilize only to a constant. If the requirement of positivity of the initial functional is abandoned, the existence of a spherical limiting mean is no longer a necessary condition for stabilization of the solution.

For definitions and notation used in the paper, see [1, 2]. Let $\Omega_\rho(x)$ be the ball in R^n of radius ρ with center at the point x ; c_n the volume of the unit ball in R^n ; $\chi_\rho(x)$ the characteristic function of the ball, equal to $(c_n \rho^n)^{-1}$ when $x \in \Omega_\rho(0)$, and to zero outside $\Omega_\rho(0)$; $u_0(x) \in S'_{1/2}(R^n)$; then, if there exists

$$\lim_{\rho \rightarrow +\infty} \chi_\rho(x) * u_0(x) = M_{u_0(x)}(x) \quad \text{in } S'_{1/2}(R^n), \quad (3)$$

we shall call this limit the generalized spherical limiting mean of the functional $u_0(x)$.

It follows easily from the results of [1] that the solution of the Cauchy problem (1)–(2) can be written in the form

$$u(x, t) = \left(u_0(y), \frac{1}{(2\sqrt{\pi t})^n} \exp \left[-\frac{|x-y|^2}{4t} \right] \right), \quad t > 0, \quad (4)$$

and the space $S'_{1/2}(R^n)$ is a class of existence and uniqueness.

Theorem 1. Let $u_0(x)$ be continuous and satisfy the estimate: for every $\varepsilon > 0$ there exists C_ε such that

$$|u_0(x)| \leq C_\varepsilon - \exp[\varepsilon|x|^2], \quad (5)$$

and suppose that the spherical limiting mean of $u_0(x)$ exists at the point x_0 , i.e.

$$\lim_{\rho \rightarrow +\infty} (\chi_\rho(x) * u_0(x))(x_0) = \lim_{\rho \rightarrow +\infty} \frac{1}{c_n \rho^n} \int_{\Omega_\rho(0)} u_0(x_0 + \xi) d\xi = M_{u_0(x)}(x_0). \quad (6)$$

Then the solution of the Cauchy problem (1)–(2) stabilizes at the point x_0 to $M_{u_0(x)}(x_0)$,

$$\lim_{t \rightarrow +\infty} u(x_0, t) = M_{u_0(x)}(x_0).$$

Proof. In view of the linearity of the equation and of the operation of finding the spherical limit mean, without loss of generality we may assume $M_{u_0(x)}(x_0) = 0$. Since $u_0(x)$ is continuous, has estimate (5), then $u_0(x) \in S'_{1/2}(R^n)$, and, according to (4),

$$u(x_0, t) = \int \frac{u_0(y)}{(2\sqrt{\pi t})^n} \exp \left[-\frac{|x_0 - y|^2}{4t} \right] dy = \frac{1}{\pi^{n/2}} \int u_0(x_0 + 2\sqrt{t}\eta) \exp[-|\eta|^2] d\eta.$$

Passing to spherical coordinates and extracting the spherical mean, i.e. integrating by parts, where the nonintegral terms turn into zero, we have

$$u(x_0, t) = \frac{2}{\pi^{n/2}} \int_0^\infty r^{n+1} \exp[-r^2] (\chi_{2\sqrt{t}r}(x) * u_0(x))(x_0) dr.$$

Then, passing to the limit as $t \rightarrow +\infty$ under the integral sign (and this is possible, since the integrand, by virtue of (6), is bounded on the half-axis) and taking into account that $M_{u_0(x)}(x_0) = 0$, we obtain the assertion of the theorem.

Suppose now that $u_0(x) \in S'_{1/2}(R^n)$ and that $M_{u_0(x)}(x)$ exists in $S'_{1/2}(R^n)$; then the spherical limit mean of the solution of the generalized Cauchy problem (1)–(2) is itself the solution of the Cauchy problem for equation (1) with initial function equal to $M_{u_0(x)}(x)$ (the generalized spherical limit mean of the original initial condition $u_0(x)$).

Indeed,

$$\begin{aligned} \lim_{\rho \rightarrow +\infty} \chi_\rho(x) * u(x, t) &= \lim_{\rho \rightarrow +\infty} \frac{1}{c_n \rho^n} \int_{\Omega_\rho(0)} \left(u_0(y), \frac{1}{(2\sqrt{\pi t})^n} \exp \left[-\frac{|x - \xi - y|^2}{4t} \right] \right) d\xi \\ &= \lim_{\rho \rightarrow +\infty} \left(\chi_\rho(y) * u_0(y), \frac{1}{(2\sqrt{\pi t})^n} \exp \left[-\frac{|x - y|^2}{4t} \right] \right) \\ &= \left(M_{u_0(x)}(y), \frac{1}{(2\sqrt{\pi t})^n} \exp \left[-\frac{|x - y|^2}{4t} \right] \right) = M_{u(x, t)}(x). \end{aligned}$$

Comparing with (4), we derive the required assertion. Using Theorem 1 for $u(x, t)$ with initial function $u(x, t_0)$ for some t_0 , we obtain

$$\lim_{t \rightarrow +\infty} u(x, t) = M_{u(x, t_0)}(x).$$

Since t_0 may be chosen arbitrarily, $M_{u(x, t)}(x)$ does not depend on t and satisfies equation (1); consequently, $M_{u(x, t)}(x)$ is a harmonic function in the generalized sense, and then it is an ordinary harmonic function (see (2)). Thus, the following is true.

Theorem 2. Let $u_0(x) \in S'_{1/2}(R^n)$ and let the generalized spherical limit mean (3) exist; then it is a harmonic function.

Corollary. In order that $u_0(x) \in S'_{1/2}(R^n)$ be a harmonic function in R^n , it is necessary and sufficient that

$$\lim_{\rho \rightarrow +\infty} \chi_\rho(x) * u_0(x) = u_0(x). \quad (7)$$

Let $u_0(x) \in S'_{1/2}(R^n)$ and be positive; then it is given by a positive measure

$$(u_0(x), \varphi(x)) = \int \varphi(x) d\mu_{u_0}(x), \quad \varphi(x) \in S_{1/2}(R^n), \quad (8)$$

where the measure satisfies the condition

$$\int \exp[-a|x|^2] d\mu_{u_0}(x) < +\infty \quad (9)$$

for any $a > 0$, see ⁽³⁾.

We note that the spherical limiting mean of a positive measure, if it exists, is a constant.

Indeed, let

$$\lim_{\rho \rightarrow +\infty} \chi_\rho(x) * u_0(x) = \lim_{\rho \rightarrow +\infty} \frac{1}{c_n \rho^n} \int_{\Omega_\rho(x)} d\mu_{u_0}(\xi) = M_{u_0(x)}(x);$$

then $M_{u_0(x)}(x)$ is a harmonic function. It is nonnegative and, by Liouville' s theorem, constant.

In the class of positive initial functionals from $S'_{1/2}(R^n)$, the existence of the spherical limiting mean is not only sufficient for stabilization of the solution, but also necessary.

Theorem 3. Let $u(x, t)$ be the solution of the Cauchy problem (1)–(2); $u_0(x) \in S'_{1/2}(R^n)$ and positive; moreover,

$$\lim_{t \rightarrow +\infty} u(x_0, t) = M, \quad (10)$$

then $M_{u_0(x)}(x)$ exists and is equal to M .

Proof. Without loss of generality, one may assume $M = 0$. From (4), (8), and (10) we have

$$\lim_{t \rightarrow +\infty} u(x_0, t) = \lim_{t \rightarrow +\infty} \frac{1}{\pi^{n/2}} \int \frac{1}{(2\sqrt{t})^n} \exp \left[-\frac{|x_0 - y|^2}{4t} \right] d\mu_{u_0}(y) = 0. \quad (11)$$

The following estimate is valid:

$$\frac{1}{c_n \rho^n} \int_{\Omega_\rho(x_0)} d\mu_{u_0}(\xi) \leq \frac{e}{c_n \rho^n} \int_{\Omega_\rho(x_0)} \exp \left[-\frac{|x_0 - \xi|^2}{4\rho^2} \right] d\mu_{u_0}(\xi) \leq \frac{e}{c_n} \int \frac{1}{\rho^n} \exp \left[-\frac{|x_0 - \xi|^2}{4\rho^2} \right] d\mu_{u_0}(\xi).$$

Passing in it to the limit as $\rho \rightarrow +\infty$ and taking (11) into account, we obtain

$$M_{u_0(x)}(x_0) = 0.$$

Remark. Let $u_0(x)$ be continuous, satisfy estimate (5), and be bounded below. Adding a sufficiently large constant, we make the initial condition positive; therefore Theorem 3 remains valid also for such functions $u_0(x)$; this generalizes the corresponding results of the work ⁽⁵⁾.

Thus, in the class of positive initial functionals the solution of the generalized Cauchy problem (1)–(2) can stabilize only to a constant.

All the results obtained are valid for the generalized Cauchy problem for the ultraparabolic equation considered in work ⁽¹⁾.

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CITED LITERATURE

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