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Abstract

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MATHEMATICS

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GAUSSIAN STATIONARY PROCESSES WITH ASYMPTOTICALLY POWER-LAW SPECTRUM

(Presented by Academician A. N. Kolmogorov on 21 V 1968)

1. Let $\xi(\varphi)$, $\varphi(t) \in K_T$, be a real Gaussian process on the interval $[0, T]$, generalized in the sense of Gelfand–Ito, with zero mean and correlation functional $\bar{B}(\varphi, \psi) = M\xi(\varphi)\xi(\psi)$; K_T is the space of infinitely differentiable functions $\varphi(t)$ with support in $[0, T]$ and with the usual topology of L. Schwartz.

Denote by $H_t(\xi)$, $t \leq T$, the Hilbert spaces obtained by closing the linear span of the random variables $\{\xi(\varphi), \varphi \in K_t\}$ with respect to the scalar product $(\xi(\varphi), \xi(\psi)) = B(\varphi, \psi)$. The space $H_T(\xi)$ is isomorphic to the space of functionals $H_T(B) \subset K'_T$, called the space with reproducing kernel $B(\varphi, \psi)$ (see ⁽²⁾). The isomorphism $H_T(\xi) \leftrightarrow H_T(B)$ is effected by the unitary correspondence $U: H_T(\xi) \ni \eta \leftrightarrow (U\eta)(\varphi) = M\eta\xi(\varphi) \in H_T(B)$ and

$$\langle f_1, f_2 \rangle_{H_T(B)} = (U^{-1}f_1, U^{-1}f_2),$$

$f_1, f_2 \in H_T(B)$.

A number of statistical problems, such as extrapolation of the process $\xi(\varphi)$, description of measures equivalent to the measure induced by the process $\xi(\varphi)$, etc., are solved by using the spaces $H_T(\xi)$ and $H_T(B)$ (see, for example, ^(2–5)). In the present note a method is developed for the analytic description of the spaces $H_T(\xi)$ and $H_T(B)$ on the basis of the canonical representation ⁽⁶⁾ of the process $\xi(\varphi)$ and the solution of certain Wiener–Hopf type equations by a method close to that used by M. G. Krein ⁽⁷⁾ in inverse problems for an inhomogeneous string. This makes it possible to study $H_T(\xi)$ and $H_T(B)$ for processes $\xi(\varphi)$ that include, as particular cases, the Gaussian process $\xi(t)$ with stationary increments

$$M|\xi(t) - \xi(s)|^2 = \text{const} \cdot |t - s|^\alpha, \quad \alpha > 0,$$

and stationary processes with spectral density

$$c_1 < f(\lambda)(1 + |\lambda|)^\mu < c_2, \quad |\mu| < \infty.$$

A somewhat different approach is used in Sec. 5 for the study of homogeneous fields.

2. We note a simple property of the functionals from $H_T(B)$. Let $X \supset K_T$ be a linear topological space containing K_T as an everywhere dense subset, and suppose the topology in K_T induced from X is weaker than the topology of K_T ; we shall call X an extension of K_T . If the correlation functional $B(\varphi, \varphi)$ is continuous in the extension $X \supset K_T$, then $H_T(B) \subset X'$, where X' is the conjugate space to X . This follows from the Cauchy–Bunyakovsky inequality:

$$|f(\varphi)|^2 = |(U^{-1}f, \xi(\varphi))|^2 \leq \|U^{-1}f\|^2 B(\varphi, \varphi).$$

The process $\xi(\varphi)$ can then, by continuity, be extended to X . It is clear that the maximal extension of K_T is the closure of K_T in the metric $\|\varphi\|^2 = B(\varphi, \varphi)$, which we denote by $L_T^2(B)$. The extension of $\xi(\varphi)$ to $L_T^2(B)$ makes it possible to describe the elements of $H_T(\xi)$ as follows: $H_T(\xi) \ni \eta = \xi(q)$, $q \in L_T^2(B)$.

Definition. Let E_t be a family of projection operators in $H_T(\xi)$ onto the subspaces $H_t(\xi)$, $t \leq T$. Following Hida–Cramér, we shall call the process $\xi(\varphi)$, $\varphi \in K_T$, a process of multiplicity 1 if there exists an element $\eta \in H_T(\xi)$ for which the linear span $\{E_t\eta = \eta_t, 0 \leq t \leq T\}$ is dense in $H_T(\xi)$ (cf. (8)); the element η will be called cyclic.

Theorem 1. Let $\xi(\varphi)$, $\varphi \in K_T$, be a Gaussian process of multiplicity 1; $\eta \in H_T(\xi)$ a cyclic element, $\eta_t = E_t\eta = \xi(q_t)$, $q_t \in L_T^2(B) \cap$

$\eta L_t^2(B)$ and $f_t = U\eta_t \in H_t(B)$, $f_\varphi(\cdot) = B(\varphi, \cdot) \in H_T(B)$. Then $\xi(\varphi)$ admits the canonical representation

$$\xi(\varphi) = \int_0^\tau \frac{df_t(\varphi)}{d\sigma(t)} d\eta_t = \int_0^\tau \frac{df_\varphi(q_t)}{d\sigma(t)} d\eta_t, \quad \varphi \in K_\tau, \quad (1)$$

where the integral is defined as a stochastic integral with respect to the process η_t with independent increments, $\sigma(t) = M|\eta_t|^2 = B(q_t, q_t) \geq 0$ is a nondecreasing bounded function, and $df(\cdot)/d\sigma(\cdot)$ is understood as the derivative in the Radon–Nikodym sense.

Theorem 2. Let $B(\varphi, \psi)$ be the correlation functional of a generalized Gaussian process $\xi(\varphi)$, $\varphi \in K_T$, continuous in the extension $X \supset K_T$. If for some $f_0 \in X'$ and every t , $0 \leq t \leq T$, there exists an element $q_t \in X_t$, where X_t is the closure of K_t in X , such that

$$B(\varphi, q_t) = f_0(\varphi), \quad \varphi \in K_t, \quad (2)$$

then $f_0(q_t) = \sigma(t) \geq 0$ defines a nondecreasing bounded function of t , and the functions $\hat{f}(t) = f(q_t)$ for $f \in H_T(B)$ are absolutely continuous with respect to $\sigma(t)$, moreover $\hat{f}(t)$ and $\sigma(t)$ do not depend on the nonunique choice of q_t . If the linear span of $\{q_t, 0 \leq t \leq T\}$ is dense in X , then $H_T(B)$ is unitarily isomorphic to $L^2([0, T], d\sigma(t))$, and

$$\langle f_1, f_2 \rangle_{H_T(B)} = \int_0^T \frac{d\hat{f}_1(t) d\hat{f}_2(t)}{d\sigma(t)}. \quad (3)$$

3. Consider a generalized stationary Gaussian process $\xi(\varphi)$, $\varphi \in K_T$, with mean value $M\xi(\varphi) = 0$ and correlation functional $B_\mu(\varphi, \psi) = \int \tilde{\varphi}(\lambda) \overline{\tilde{\psi}(\lambda)} |\lambda|^{-\mu} d\lambda$, $\mu < 1$, where $\tilde{\varphi}(\lambda)$ is the Fourier transform of the function $\varphi(t)$.

Theorem 3. 1) The integral equation

$$\int \tilde{\varphi}(\lambda) \overline{\tilde{q}_t(\lambda)} |\lambda|^{-\mu} d\lambda = \int_0^t \varphi(t) dt, \quad \varphi \in K_t, \quad (4)$$

has the solution

$$q_t(s) = \frac{1}{2\pi} \frac{1}{\Gamma(1-\mu)} (t-s)_+^{-\mu/2} s_+^{-\mu/2}, \quad 0 < s \leq T, \quad (5)$$

($u_+ = u$ for $u \geq 0$ and $u_+ = 0$ for $u \leq 0$); the family $\{q_t(s), 0 \leq t \leq T\}$ is dense in $L^2(0, T)$.

- 2) For $0 \leq \mu < 1$, the elements of $H_T(B_\mu)$ are $f(\varphi) = \int_0^T f(t) \varphi(t) dt$, $f(t) \in L^{2/(1-\mu)}$, and

$$\langle f_1, f_1 \rangle_{H_T(B_\mu)} = \frac{1}{2\pi} \int_0^T |t^{\mu/2} D^{\mu/2} [f(t) t^{-\mu/2}]|^2 dt, \quad (6)$$

where

$$D^\alpha[f] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds$$

is the fractional differentiation operator of order α .

- 3) For $-1 \leq \mu < 0$, the elements of $H_T(B_\mu)$ are singular generalized functions of first order: $H_T(B_\mu) \ni f(\varphi) = \int_0^T F(t) \varphi'(t) dt$, $F(t) \in \text{Lip}((1+\mu)/2)$, if $-1 < \mu < 0$, and $F(t) \in L^p$, $p > 1$, if $\mu = -1$, ($\frac{d}{dt} F(t) = f(t)$); in this case

$$\langle f, f \rangle_{H_T(B_\mu)} = \frac{1}{2\pi} \int_0^T \left| t^{\mu/2} \frac{d}{dt} \int_0^t F(s) d \left[\frac{(t-s)^{-\mu/2} s^{-\mu/2}}{\Gamma(1-\mu)} \right] \right|^2 dt. \quad (7)$$

Using Theorem 3 and the method developed in (2), it is easy to write out the Radon-Nikodym derivative of the measures P_1 and P_2 corresponding to the processes $\xi(\varphi)$ and $\xi(\varphi) + m(\varphi)$, $m(\varphi) \in H_T(B_\mu)$.

The process $\xi(\varphi)$ is the derivative of order $([(1+\mu)/2] + 1)$ of the Gaussian process $x(t)$ with homogeneous increments, invariant with respect to the similarity transformation (8). This observation makes it possible to transfer Theorem 3 also to the process $x(t)$. Let $|\mu| < 1$, and let $x(t)$ be a Gaussian process with correlation function

$$R_\mu(s, t) = k(|s|^{1+\mu} + |t|^{1+\mu} - |t-s|^{1+\mu}), \quad k = 2 \sin \frac{\mu}{2} \pi \Gamma(-1-\mu).$$

Since $x(t) = \xi(\chi_{0t})$, $H_T(R_\mu)$ consists of functions $F(t) = f(\chi_{0t})$, $f(\varphi) \in H_T(B_\mu)$, with scalar product

$$\langle F_1(t), F_2(t) \rangle_{H_T(R_\mu)} = \langle f_1, f_2 \rangle_{H_T(B_\mu)}.$$

Combining Theorem 1 with Theorem 3, we obtain the canonical representation of the process $x(t)$ in the form

$$x(t) = \int_0^t Q(t, s) d\eta(s),$$

where

$$Q(t, s) = \begin{cases} \int_s^t (\tau - s)^{\mu/2-1} \tau^{\mu/2} d\tau, & \text{for } 0 < \mu < 1, \\ \frac{1}{\pi} \sin \frac{\mu\pi}{2} \left\{ \int_s^t (\tau - s)^{\mu/2-1} [\tau^{\mu/2} - s^{\mu/2}] d\tau + \frac{2}{\mu} (ts - s^2)^{\mu/2}, \right. & \text{for } -1 \leq \mu < 0, \end{cases}$$

and the process with independent increments $\eta(s)$, for which $H_t(x) = H_t(\eta)$, $t \leq T$, has the form

$$\eta(s) = \begin{cases} \int_0^s (s-t)^{-\mu/2} t^{-\mu/2} dx(t), & \text{for } 0 < \mu < 1, \\ \int_0^s x(t) d[(s-t)^{-\mu/2} t^{-\mu/2}], & \text{for } -1 \leq \mu \leq 0. \end{cases}$$

Hence, in the usual way, one obtains formulas for prediction of $x(t)$, first obtained in (9).

4. Let $\xi_i(\varphi)$, $\varphi \in K_T$, be Gaussian stationary processes with means $m_i(\varphi) = 0$, correlation functions

$$B_i(\varphi, \varphi) = \psi_i(\varphi * \varphi^*), \quad \psi_i \in K',$$

$\varphi^*(x) = \varphi(-x)$, and spectral measures $F_i(d\lambda)$ (1); P_i are the measures induced by the processes $\xi_i(\varphi)$, $\varphi \in K_T$, in the space K'_T , $i = 1, 2$.

Theorem 4. If $F_i(d\lambda)$ are absolutely continuous and

$$0 < c_1 < \frac{F_1(d\lambda)}{d\lambda}(1 + |\lambda|)^{2n+\mu} < c_2 \quad \text{as } \lambda \rightarrow \infty,$$

where n is an integer and $-1 \leq \mu < 1$, then the measures P_i are equivalent if and only if

$$\Delta^{(2n)}(\varphi * \psi^*) \in H_T(B_\mu) \otimes H_T(B_\mu),$$

where

$$\Delta(\varphi) = \psi_1(\varphi) - \psi_2(\varphi), \quad \Delta^{(k)}(\varphi) = \Delta(\varphi^{(k)}),$$

in other words:

- 1) for $0 \leq \mu < 1$,

$$\Delta^{(2n)}(\varphi * \psi^*) = \int_0^T \int_0^T \Delta^{(2n)}(s-t) \varphi(s) \psi(t) ds dt,$$

where

$$\Delta^{(2n)}(s-t) \in L^{2/(1-\mu)}([0, T] \times [0, T]),$$

and

$$\int_0^T \int_0^T (st)^\mu \left| D_t^{\mu/2} D_s^{\mu/2} [\Delta^{(2n)}(s-t)(st)^{-\mu/2}] \right|^2 ds dt < \infty;$$

- 2) for $-1 \leq \mu < 0$, the functional

$$\Delta^{(2n-2)}(\varphi) = \int_0^T \Delta^{(2n-2)}(t) \varphi(t) dt,$$

$$\Delta^{(2n-2)}(t) \in C[0, T]$$

and

$$\int_0^T \int_0^T (st)^\mu \left| \frac{\partial^2}{\partial s \partial t} \int_0^s \int_0^t \Delta^{(2n-2)}(u-v) du dv [(s-u)(t-v)uv]^{-\mu/2} \right|^2 ds dt < \infty.$$

The proof is based on the results of (2,10).

Theorem 5. Let $f(\lambda) \geq 0$ be a locally summable function and $c_1 < f(\lambda)|\lambda|^{(2n+\mu)} < c$ as $\lambda \rightarrow \infty$, $-1 \leq \mu < 1$, n an integer. For the existence of a solution of the Wiener-Hopf integral equation

$$a(t) = \int e^{i\lambda t} \varphi(\lambda) f(\lambda) d\lambda, \quad 0 \leq t \leq T,$$

in the class of functions $L_T^2(f)$ —the closure of $\{\varphi(\lambda), \tilde{\varphi}(t) \in K_T\}$, in the metric

$$\int |\varphi|^2 f d\lambda = \|\varphi\|_{L_T^2(f)}^2,$$

it is necessary and sufficient that $a(t)$ have $n-1$ derivatives and that $a^n(t) \in H_T(B_\mu)$; the n -th derivative is understood as a generalized one.

Theorems 4 and 5 in the case $\mu = 0$ were obtained by Yu. A. Rozanov ^(11,12).

5. Let $\xi(\varphi)$, $\varphi \in K(E^n)$, be a Gaussian homogeneous field with correlation functional

$$B(\varphi, \varphi) = \int_{E^n} |\tilde{\varphi}(\lambda)|^2 f(\lambda) d^n \lambda,$$

and let Ω be a bounded domain with smooth boundary in E^n . The notation $H_\Omega(B)$ has the same meaning as in the one-dimensional case.

Theorem 6. Let $0 < c_1 < f(\lambda)(1 + |\lambda|^2)^{-\mu} < c_2$, $\mu > 0$; then the space $H_\Omega(B)$, up to equivalence of norms, coincides with the Sobolev-Slobodetskii space $W_2^\mu(\Omega)$, whose norm has the form:

$$\|f\|_{W_2^\mu(\Omega)}^2 = \|f\|_{L^2}^2 + \sum_{|\alpha|=[\mu]} \|f^{(\alpha)}\|_{L^2}^2 + \sum_{|\alpha|=[\mu]} \int_{\Omega \times \Omega} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|^2}{|x - y|^{n+2\beta}} d^n x d^n y,$$

where

$$\beta = \mu - [\mu], \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \sum_{i=1}^n \alpha_i$$

and

$$f^\alpha(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f(x)$$

is a generalized derivative in the sense of Sobolev ⁽¹³⁾.

Theorem 6 follows from results of V. M. Babich and L. N. Slobodetskii (see ⁽¹³⁾).

Remark 1. Theorems 4 and 5, taking Theorem 6 into account, admit obvious reformulations for the case of homogeneous fields, one of which has the spectrum indicated above.

Remark 2. Since

$$W_2^\mu(\Omega) \otimes W_2^\mu(\Omega) \supset W_2^{2\mu}(\Omega \times \Omega),$$

from Theorem 4 we obtain sufficient conditions for the equivalence of two homogeneous fields, namely,

$$\Delta(s - t) \in W_2^{2\mu}(\Omega \times \Omega).$$

Remark 3. Comparing Theorems 3, 4, and 6 in the case $E^n = E^1$ makes it possible to describe the space $W_2^\mu([0, T])$, $\mu > 0$, in a new way.

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REFERENCES

1. I. M. Gel'fand, N. Ya. Vilenkin, *Some Applications of Harmonic Analysis. Generalized Functions*, M., 1961.
2. Yu. I. Golosov, A. M. Yaglom, Abstracts of scientific communications, International Congress of Mathematicians, Section II, M., 1967.
3. J. Hajek, Czechosl. Math. J., **12**, 404 (1962).
4. E. Parzen, In: *Time Series Analysis*, Ch. II, N. Y., 1963.
5. G. M. Molchan, *Theory of Probability and Its Applications*, **12**, 747 (1967).
6. H. Cramér, *Theory of Probability and Its Applications*, **9** (1964).
7. M. G. Krein, DAN, **100**, 413 (1955).
8. A. M. Yaglom, Matem. sbornik, **37** (79), 141 (1955).
9. S. V. Grigor'ev, Uch. zap. Kazan. Univ., **125**, No. 6, 106 (1966).
10. D. S. Apokolov, *Theory of Probability and Its Applications*, **12**, 698 (1967).
11. Yu. A. Rozanov, *Theory of Probability and Its Applications*, **11**, 170 (1966).

12. Yu. A. Rozanov, *Theory of Probability and Its Applications*, **8**, 241 (1963).
13. L. N. Slobodetskii, Uch. zap. Leningrad Ped. Inst. im. A. I. Herzen, **197**, 54 (1958).

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