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Abstract

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MATHEMATICS

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ON INDECOMPOSABLE SPACES AND TOPOLOGICALLY SELF-DENSE ULTRAFILTERS

(Presented by Academician P. S. Aleksandrov, 21 IV 1969)

The basic notion in this note is that of a d -end of a topological space (Definition 1). With its help, in the first part we obtain a generalization of Šoke's theorem on topologically self-dense ultrafilters (Theorem 1), and in the second — the existence of connected H -closed k -decomposable, $1 \leq k < \aleph_0$, but not $(k + 1)$ -decomposable spaces (Theorem 7), and the coabsoluteness of an arbitrary self-dense Hausdorff space with a minimal space (Theorem 9). We also note Theorem 2, which makes it possible to obtain minimal relaxations of spaces by means of centered systems, and Corollary 2: a θ -homeomorphism preserves extremal disconnectedness. Finally, in the third part the space of d -ends of a given topological space X is constructed and its relation to the space $\theta(X)$ of all ends of the space X is clarified.

Definition 1. A system of subsets of a topological space X is called d -centered* if the intersection of any finite family of its elements is dense in X .

A d -centered system of subsets of the space X is called a d -end of the space X if it is not contained in any distinct d -centered system of subsets of X . We shall denote the set of all d -ends of the space X by dX .

It is easily proved that every d -centered system of subsets of the space X is contained in at least one d -end of the space X . Put

$$d_M = \{A \cap M : A \in d\}.$$

Proposition 1. 1°. Let $d \in dX$ and $M \subseteq X$. Then there exist disjoint canonical open subsets U and G in X such that

$$[U \cup G] = X, \quad M \cap U \in d_U,$$

and there exists $A \in d$ for which

$$A \cap M \cap G = \Lambda.$$

2°. Let d and d' be two d -ends of the space X . Then there exist disjoint canonical open subsets U and G in X such that

$$[U \cup G] = X, \quad d_U = d'_U,$$

and there exist $A \in d$ and $A' \in d'$ for which

$$A \cap A' \cap G = \Lambda.$$

I. Šoke (¹⁴) introduced the notion of a topologically self-dense ultrafilter (this is an ultrafilter in the set of points of the segment $[0, 1]$ with a base of self-dense sets) and, under the assumption of the continuum hypothesis, proved the existence of such an ultrafilter. Below we give three proofs of the existence of ultrafilters with a base of self-dense sets in every self-dense T_0 -space, not relying on the continuum hypothesis.

Lemma 1. Let** $p \in \theta(X)$ and $d \in dX$, where X is a nonempty topological space. Then

$$p \wedge d = \{U \cap A : U \in p, A \in d\}$$

is a base of an ultrafilter in the set X .

* Such systems were considered in (³), p. 121.

** An end of the space X is a maximal centered system of open subsets in X . The space $\theta(X)$ is the set of all ends of the space X , endowed with the topology whose base is formed by the sets

$$O_U = \{p \in \theta(X) : p \ni U\}.$$

In this topology $\theta(X)$, as shown by S. Iliadis and S. V. Fomin (⁷), is an extremally disconnected bicomactum.

Put $tX = \min\{sU : U \neq \Lambda \text{ and is open in } X\}^*$. From Lemma 1 it easily follows

Theorem 1. *In every nonempty crowded T_0 -space X ($tX \neq 1$) there exists an ultrafilter with a base each element of which is a crowded subset of X of cardinality tX , whose closure is a canonical closed set in X .*

Hence, by virtue of the well-known theorem of P. S. Aleksandrov stating that every zero-dimensional perfect compactum (i.e., discontinuum) is homeomorphic to the Cantor perfect set $\mathcal{C}$, we immediately obtain

Corollary 1. *There exists an ultrafilter in $\mathcal{C}$ (and, consequently, in $[0, 1]$) with a base each element of which is a crowded countable set whose closure is homeomorphic to $\mathcal{C}$.*

This is the first half of Ślupecki's theorem ((¹⁴), Theorem 10), proved by him under the assumption of the continuum hypothesis.

Second proof. We shall call a system of subsets of a space X q -centered if the intersection of every nonempty finite family of its elements is a nonempty crowded subset of X . A maximal q -centered system of subsets of X will be called a q -end of the space X . A space Y will be called a q -relaxation of the space X if Y is the set X , endowed with the topology with prebase $\mathcal{T} \cup q$, where $\mathcal{T}$ is the topology of the space X and q is a q -centered system of subsets of X .

Theorem 2. *Let X be a crowded space and let q be a q -end in X . Then the q -relaxation of the space X is a minimal** space in which q is an end.*

Since a minimal T_0 -space is*** SI and since every end in a nonempty SI , as shown in (5), forms a base of an ultrafilter, it follows from Theorem 2 that

Theorem 3. Every q -end of a nonempty crowded T_0 -space X is a base of an ultrafilter in the set X .

The third proof is simply a generalization of the second: let X be a nonempty crowded T_0 -space; let Y be a minimal relaxation of the space X , whose existence was proved by Hewitt (13) and Katětov (8); then, as was already noted above, every end in Y is a base of an ultrafilter in the set Y and, consequently, in the set X , and every set from this end is, obviously, crowded in the space X .

II. **Definition 2.** A space Y is called a d -relaxation of the space X if Y is the set X , endowed with the topology with prebase $\mathcal{T} \cup d$, where $\mathcal{T}$ is the topology of the space X and d is a d -centered system of subsets of X .

Theorem 4. Let Y be a relaxation of the space X . Then the following assertions are equivalent: (1) X and Y have one and the same supply of canonical open sets, or, equivalently, one and the same supply of canonical closed sets. (2) X and Y have one and the same associated semiregular space**. (3) Y is a d -relaxa-

* $sX = \min\{|A| : A \text{ is crowded in } X\}$.

** A space Y is called a relaxation of the space X if Y consists of the same points as X , and if the identity mapping of Y onto X is continuous. A space is called minimal if it is crowded, and every one of its proper relaxations contains isolated points.

*** One says that X is SI if X is crowded and if every nonempty crowded subset of X is irresolvable, i.e., does not contain two disjoint dense subsets.

**** The canonical open subsets of the space X form a base of a semiregular topology on the set X . The set X , endowed with this topology, is called the semiregular space associated with X (see (3), p. 121).

spaces X . (4) The identity mapping X onto Y is θ -discontinuous*.

From Theorem 4 a number of important properties of d -relaxations follow easily.

Proposition 2. A d -relaxation of a connected space is connected.

Proposition 3. Every d -relaxation of an extremally disconnected space is extremally disconnected. Conversely, if a space has at least one extremally disconnected d -relaxation, then the space itself is extremally disconnected.

Proposition 4. A d -relaxation of an H -closed space is H -closed.

The last proposition follows from the fact that, as was proved by S. V. Fomin⁽¹²⁾, the θ -continuous image of an H -closed space is H -closed.

Proposition 5. Every proper d -relaxation of an arbitrary topological space is non-semiregular (and hence irregular).

Lemma 2. A θ -homeomorphism between two semiregular spaces is a homeomorphism.

Theorem 5. The following assertions concerning topological spaces X and Y consisting of the same points are equivalent:

- (1) X and Y have one and the same stock of canonical open sets.
- (2) X and Y have one and the same associated semiregular space.
- (3) X and Y are d -relaxations of one and the same (semiregular) space.
- (4) X and Y are identically θ -homeomorphic.

Corollary 2. If two spaces are θ -homeomorphic and one of them is extremally disconnected, then the other space is also extremally disconnected.

Theorem 6. Let X be a dense-in-itself T_1 -space and $d \in dX$. Then the d -relaxation of the space X is an MI -space**.

At the seminar of P. S. Aleksandrov, B. A. Efimov posed the following problem: do there exist connected Hausdorff k -decomposable*** spaces, $2 \leq k < \aleph_0$, which are not $(k+1)$ -decomposable? Its solution is given by

Theorem 7. Every connected H -closed k -decomposable space, $1 \leq k < \aleph_0$, has a connected H -closed k -decomposable relaxation that is not $(k+1)$ -decomposable****.

Theorem 8. Every dense-in-itself extremally disconnected H -closed space has an H -closed minimal relaxation*****.

Theorem 9. Every dense-in-itself Hausdorff space is coabsolute***** with some minimal space.

* A mapping $f : X \rightarrow Y$ is called θ -continuous in the sense of S. V. Fomin⁽¹²⁾ if, for every point $x \in X$ and for every neighborhood Ofx of its image, there exists a neighborhood Ox of this point such that $f[Ox] \subseteq [Ofx]$. A one-to-one mapping that is θ -continuous in both directions is called a θ -homeomorphism.

** A space is called an MI -space if it is dense in itself and if every subset dense in it is open.

*** A space is k -decomposable if it contains k pairwise disjoint subsets dense in it ($k \geq 1$).

**** Hewitt's problem ⁽¹³⁾—whether there exist connected Hausdorff indecomposable spaces—was solved by Padmavally ⁽⁹⁾ by the construction of an example. Then Bourbaki ⁽³⁾, p.181 and Anderson ⁽²⁾ gave a proof of the existence of a connected indecomposable relaxation for every connected Hausdorff space. The existence of k -decomposable Hausdorff spaces, $2 \leq k < \aleph_0$, which are not $(k+1)$ -decomposable, was proved in ⁽⁴⁾.

***** Dense-in-itself extremally disconnected H -closed spaces exist: for every dense-in-itself Hausdorff space X , such a space is $\theta(X)$. The existence of H -closed minimal spaces was proved by Katětov ⁽⁸⁾.

***** Two Hausdorff spaces are coabsolute if they have one and the same absolute, i.e. the maximal irreducible θ -continuous preimage. For the first time, the (spectral) theory of absolutes of paracompact Hausdorff spaces was constructed by V. I. Ponomarev ⁽¹⁰⁾. Then, by the method of centered systems, S. Iliadis ⁽⁶⁾ and, by the spectral method, V. I. Ponomarev ⁽¹¹⁾ constructed it already for arbitrary Hausdorff spaces.

III. **Definition 3.** Let X be an arbitrary topological space. The space dX is the set dX , endowed with the topology whose base is formed by the sets

$$O_A = \{d \in dX : d \supset A\},$$

where $A \subset X$.

Proposition 6. For every space X , the space dX is zero-dimensional and Hausdorff (and hence completely regular).

Theorem 10. Let $\{U_\lambda : \lambda \in \mathcal{L}\}$ be a family of pairwise disjoint open subsets of the space X , whose union is dense in X . Then dX is homeomorphic to the product

$$\prod \{dU_\lambda : \lambda \in \mathcal{L}\},$$

endowed with the box topology.

Denote by $R(X)$ the maximal open decomposable subspace of the space X (such exists).

Corollary 3. For every space X , the space dX is homeomorphic to $dR(X)$.

Proposition 7. If X is a nonempty decomposable Hausdorff space, then dX is not bicomact.

The construction, due to P. S. Aleksandrov ⁽¹⁾, of the Stone-Čech extension of an arbitrary completely regular space as applied to a discrete space T gives the following: βT is the set of all ultrafilters in T with the topology whose base is formed by the sets

$$O_E = \{\mathfrak{F} \in \beta T : \mathfrak{F} \supset E\},$$

where $E \subset T$. Thus, Lemma 1 defines a natural mapping

$$\mu : \theta(X) \times dX \rightarrow \beta T,$$

where T is the set X , endowed with the discrete topology. Namely: $\mu(p, d)$ is the ultrafilter with base $p \wedge d$.

Theorem 11. The mapping μ is continuous and open, and for each $d \in dX$ the mapping μ on the set $\theta(X) \times \{d\}$ is a homeomorphism.

****Corollary 4 (B. A. Efimov)*.**** Every regular extremally disconnected space X is topologically contained in βT , where T is a discrete space of cardinality

$$s\beta X = s\theta(X) \leq sX.$$

Corollary 5. Every infinite extremally disconnected bicompactum (in particular βN , and in general βT , where T is an infinite discrete space) contains an indecomposable space**.

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* Unpublished.

** A space is indecomposable if it is dense in itself and is not decomposable.

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