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Abstract

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MATHEMATICS

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ON A GENERALIZATION OF CAUCHY' S FUNDAMENTAL THEOREM

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Consider an associative finite-dimensional algebra $\mathfrak{A}$ over the field of real numbers. Let $e_1, e_2, \dots, e_s$ be some basis of $\mathfrak{A}$.

An arc Γ in $\mathfrak{A}$ is called rectifiable if it can be given parametrically by

$$y(t) = \sum_{i=1}^s y_i(t) e_i, \quad t_0 \leq t \leq t_1,$$

where the $y_i(t)$ are continuous and of bounded variation. Let $f(y)$ be a continuous mapping of $\mathfrak{A}$ into $\mathfrak{A}$. The sum

$$\sum_i f(y(\xi_i))(y(t_{i+1}^{(n)}) - y(t_i^{(n)})),$$

where ξ_i is an arbitrary point of $[t_{i+1}^{(n)}, t_i^{(n)}]$, as

$$\max_i (t_{i+1}^{(n)} - t_i^{(n)}) \rightarrow 0$$

tends to a limit independent of the sequences $\{t_i^{(n)}\}$. We shall call this limit, as usual, the integral of $f(y)$ and denote it by

$$\int_{\Gamma} f(y) dy.$$

The linear span of the elements $xy - yx = [x, y]$ for $x, y \in \mathfrak{A}$ is called the commutator of $\mathfrak{A}$ and is denoted by $[\mathfrak{A}]$. Let Γ be a rectifiable contour in $\mathfrak{A}$. Then the following holds.

Theorem 1. *For any integer $n \geq 0$,*

$$\int_{\Gamma} y^n dy \in [\mathfrak{A}].$$

Proof. As in the classical case ⁽¹⁾, it is enough to prove Theorem 1 for a triangular contour. Let Γ be a triangular contour with vertices A, B, O . The vertex O lies at the zero of the algebra. Then

$$\int_{\Gamma} y^n dy = \int_{OA} y^n dy + \int_{AB} y^n dy + \int_{BO} y^n dy.$$

Put $\overline{OA} = a$, $\overline{AB} = b$, $\overline{BO} = d$.

Clearly, $a + b + d = 0$. Denote by $M(n - m, m)$ the set of monomials α in the expansion of $(a + b)^n$ that contain m factors b and $n - m$ factors a , and put

$$Q(n - m, m) = \sum_{\alpha \in M(n - m, m)} \alpha.$$

Then

$$\begin{aligned} \int_{OA} y^n dy &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \left(\frac{i}{N} a \right)^n \left(\frac{1}{N} a \right) = a^{n+1} \int_0^1 t^n dt = \frac{1}{n+1} a^{n+1}, \\ \int_{AB} y^n dy &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \left(a + \frac{i}{N} b \right)^n \left(\frac{1}{N} b \right) = \left[\sum_{m=0}^n \frac{1}{m+1} Q(n - m, m) b \right], \\ \int_{BO} y^n dy &= \lim_{N \rightarrow \infty} \sum_{i=1}^N \left\{ (a + b) + \frac{i}{N} d \right\}^n \left(\frac{1}{N} d \right) = -(a + b)^{n+1} \lim_{N \rightarrow \infty} \sum_{i=1}^N \left(1 - \frac{i}{N} \right)^n \frac{1}{N} \\ &= -(a + b)^{n+1} \int_0^1 (1 - t)^n dt = -\frac{1}{n+1} (a + b)^{n+1}. \end{aligned}$$

After simplification we obtain

$$\int_{\Gamma} y^n dy = \sum_{m=0}^{n-1} \left\{ \frac{1}{m+1} Q(n - m, m) b - \frac{1}{n+1} Q(n - m, m + 1) \right\}.$$

We shall show that the difference

$$\rho = (n + 1) Q(n - m, m) b - (m + 1) Q(n - m, m + 1) \in [\mathfrak{A}], \quad m = 0, \dots, n - 1.$$

The number of terms of each sum

$Q(n - m, m) b$ and $Q(n - m, m + 1)$ are equal respectively to $\binom{n}{m}$ and $\binom{n+1}{m+1}$. Therefore the number of monomials in $(n + 1) Q(n - m, m) b$ and in $(m + 1) Q(n - m, m + 1)$ is the same. We shall match the monomials from these sums so that their difference is a commutator.

Let α be a monomial. Then α_i is equal to the product of the i first factors of α in the order in which they occur, and β_j , respectively, to the product of the j last factors, i.e. $\alpha = \alpha_i \beta_{n-i}$, if α has n factors. Let, further,

$$R_i(\alpha) = \alpha_i b \beta_{n-i}, \quad R_i(\alpha + \alpha' + \dots) = R_i(\alpha) + R_i(\alpha') + \dots$$

Lemma 1.

$$\begin{aligned} R(Q(n, m)) &= bQ(n, m) + \sum_{i=1}^{n+m-1} R_i(Q(n, m)) + Q(n, m)b = \\ &= (m+1)Q(n, m+1). \end{aligned} \quad (1)$$

Proof. If (1) holds for $Q(n-1, m)$ and $Q(n, m-1)$, then it also holds for $Q(n, m)$. Indeed,

$$Q(n, m) = Q(n-1, m)a + Q(n, m-1)b. \quad (2)$$

Then

$$R(Q(n, m)) = R(Q(n-1, m)a) + R(Q(n, m-1)b).$$

But

$$R(Q(n-1, m)a) = R(Q(n-1, m))a + Q(n-1, m)ab$$

and

$$R(Q(n, m-1)b) = R(Q(n, m-1))b + Q(n, m-1)b^2.$$

Taking now (1) and (2) into account, we obtain

$$\begin{aligned} R(Q(n, m)) &= (m+1)Q(n-1, m+1)a + mQ(n, m)b \\ &\quad + [Q(n-1, m)a + Q(n, m-1)b]b \\ &= (m+1)Q(n-1, m+1)a + mQ(n, m)b + Q(n, m)b \\ &= (m+1)[Q(n-1, m+1)a + Q(n, m)b] \\ &= (m+1)Q(n, m+1). \end{aligned}$$

To prove the lemma it remains to show that (1) holds for $Q(0, m)$ and $Q(n, 0)$ for arbitrary n and m . But from the definition of R we have

$$R(Q(0, m)) = (m+1)b^{m+1} = (m+1)Q(0, m+1),$$

$$R(Q(n, 0)) = ba^n + aba^{n+1} + \dots + a^n b = Q(n, 1).$$

Put now $\pi_j(\alpha) = \beta_{n-j}\alpha_j$ for $\alpha \in M(n-m, m) = M$,

$$\pi_j(\alpha + \alpha' + \dots) = \pi_j(\alpha) + \pi_j(\alpha') + \dots$$

Since from $\pi_j(\alpha) = \pi_j(\alpha')$ it follows that $\alpha = \alpha'$ and conversely (here equality is understood in the sense of coincidence of the factors in the same positions), we have

$$\pi_j \left(\sum_{\alpha \in M} \alpha \right) = \sum_{\alpha \in M} \pi_j(\alpha) = \sum_{\alpha \in M} \beta_{n-j} \alpha_j = Q(n-m, m).$$

Therefore

$$\begin{aligned} \pi_j(Q(n-m, m)b) &= \sum_{\alpha \in M} \pi_j(\alpha b) = \sum_{\alpha \in M} \beta_{n-j} b \alpha_j = \\ &= R_{n-j} \left(\sum_{\alpha \in M} \beta_{n-j} \alpha_j \right) = R_{n-j}(Q(n-m, m)). \end{aligned} \quad (3)$$

By virtue of the lemma and (3) we have

$$\begin{aligned} (m+1)Q(n-m, m) &= bQ(n-m, m) + \sum_{i=1}^{n-1} R_i(Q(n-m, m)) + \\ &+ Q(n-m, m)b = bQ(n-m, m) + \sum_{j=1}^{n-1} \pi_j(Q(n-m, m)b) + Q(n-m, m)b. \end{aligned}$$

Consequently

$$\begin{aligned} \varphi &= Q(n-m, m)b - bQ(n-m, m) + \sum_{j=1}^{n-1} \{Q(n-m, m)b - \pi_j(Q(n-m, m)b)\} = \\ &= \sum_{\alpha \in M} [\alpha, b] + \sum_{j=1}^{n-1} \left(\sum_{\alpha \in M} (\alpha b - \pi_j(\alpha b)) \right). \end{aligned}$$

Since from the definition of π_j it follows that

$$\alpha b - \pi_j(\alpha b) = [\alpha_j, \beta_{n-j} b],$$

then

$$\rho = \sum_{\alpha \in M} [\alpha, b] + \sum_{j=1}^{n-1} \left(\sum_{\alpha \in M} [\alpha_j, \beta_{n-j} b] \right) \in [\mathfrak{A}].$$

The theorem is proved.

Considering $[\mathfrak{A}]$ as a linear subspace of $\mathfrak{A}$, define B , $B \perp [\mathfrak{A}]$. If $L = a_1 e_1 + \dots + a_s e_s \perp [\mathfrak{A}]$, then for $x = x_1 e_1 + \dots + x_s e_s \in [\mathfrak{A}]$ we have $L(x) = a_1 x_1 + \dots + a_s x_s = 0$. The totality of such forms has dimension equal to the dimension of B . According to the theorem, the integrals $\int_a^b y^n dy$, taken along different paths from a to b , differ by an element of $[\mathfrak{A}]$. Therefore the following generalization of the fundamental theorem of Cauchy holds.

Theorem 2. *The value*

$$L \left(\int_a^b y^n dy \right)$$

does not depend on the path connecting the points a and b .

Suppose now that $\mathfrak{A}$ has a unit. We shall assume that the basis element e_1 is the unit of $\mathfrak{A}$.

Theorem 3.

$$L \left(\int_{\Gamma} y^{-1} dy \right) = 0,$$

if the contour Γ belongs to some simply connected domain D in which y^{-1} is defined.

Proof. Since D is simply connected, for arbitrary $d > 0$ there exist in D contours Γ_i of diameter $< d$ such that

$$\int_{\Gamma} y^{-1} dy = \sum_i \int_{\Gamma_i} y^{-1} dy.$$

Let $y_i \in D$ be such that $|x - y_i| \leq d$ for $x \in \Gamma_i$, and let d be chosen so that from $|y| < d$ it follows that

$$(e_1 + y)^{-1} = \sum_{n=0}^{\infty} (-y)^n.$$

Then

$$\int_{\Gamma_i} y^{-1} dy = \int_{\Gamma_i - y_i} (y_i + z)^{-1} dz = \int_{\Gamma_i - y_i} (y_i (e_1 + y_i^{-1} z))^{-1} dz = \int_{y_i^{-1}(\Gamma_i - y_i)} (e_1 + y)^{-1} dy.$$

Since for $|y| < d$

$$(e_1 + y)^{-1} = \sum_{n=0}^{\infty} (-y)^n,$$

Theorem 2 is fulfilled for it. Theorem 3 is proved.

The function

$$\int_{e_1}^x y^{-1} dy$$

has an application in probability theory on finite groups.

Lemma 2.

$$L \left(\int_{e_1}^x y^{-1} dy \right) = \varphi(x)$$

is a one-dimensional representation of a neighborhood of the unit of $\mathfrak{A}$.

Indeed, since

$$\int_{x_1}^{x_1 x_2} y^{-1} dy = \int_{e_1}^{x_2} (x_1 y)^{-1} d(x_1 y) = \int_{e_1}^{x_2} y^{-1} dy,$$

we have

$$\varphi(x_1 x_2) L \left(\int_{e_1}^{x_1 x_2} y^{-1} dy \right) = L \left(\int_{e_1}^{x_1} y^{-1} dy + \int_{x_1}^{x_1 x_2} y^{-1} dy \right) = \varphi(x_1) + \varphi(x_2).$$

Let now $\mathfrak{A}$ be the group algebra of some group G . Then one may assume that the basis $e_1, e_2, \dots, e_s$ is the group G . In this case compute

$$\varphi_j(x) = L_j \left(\int_{e_1}^x y^{-1} dy \right),$$

where L_j , $j = 1, \dots, r$, is a basis of linear forms,

corresponding to some basis $\{\gamma^{(j)}\}$ of the orthogonal complement to $\mathfrak{A}$. It is not difficult to show that in the case of a group algebra one may take for $\{\gamma^{(j)}\}$ vectors consisting of zeros and ones, the ones in $\gamma^{(j)}$ occurring only in those places α for which e_α belongs to the j -th class of conjugate elements of G . Here a fixed numbering of these classes is meant; denote them by T_j , $j = 1, \dots, r$. Then

$$\varphi_i(x) = \sum_{j=1}^r \frac{h_j}{s} \chi_i(T_j^{-1}) \ln |R_i|, \quad (4)$$

where $|R_i|$ is the determinant of the i -th irreducible part of the group matrix of the element x ; χ_i is the character of the i -th irreducible representation; h_j is the number of elements of T_j ; s is the order of the group G .

Let $y(t)$, $t \geq 0$, $y(0) = e_1$, be a continuous curve in $\mathfrak{A}$ such that for each t there exists $y^{-1}(t)$, and for the elements $y^{-1}(t_1)y(t_2)$, denoted by $y(t_1, t_2) = p_1(t_1, t_2)e_1 + \dots + p_s(t_1, t_2)e_s$, one has $p_i(t_1, t_2) \geq 0$,

$$\sum_{i=1}^s p_i(t_1, t_2) = 1.$$

Then $y(t_1, t_2)$ may be regarded as a probability distribution on the group G . From the distributions $y(t)$ one can construct a process with independent increments on the group G such that the distribution of the increment of the process on the interval $[t_1, t_2]$ is equal to $y(t_1, t_2)$. The number of jumps of such a process is finite almost everywhere. Using expression (1) from (2), one can show that the curve $y(t)$ on $[0, t]$ is rectifiable and, for

$$\Delta_n = \max_i (t_{i+1}^{(n)} - t_i^{(n)}) \rightarrow 0, \quad \sum_{i=1}^{n-1} p_j(t_i^{(n)}, t_{i+1}^{(n)}) \rightarrow m_j(t) \geq 0, \quad j \geq 2,$$

$$\sum_{i=1}^{n-1} \{p_i(t_i^{(n)}, t_{i+1}^{(n)}) - 1\} \rightarrow m_1(t).$$

The functions $m_j(t)$, $j \geq 2$, are nondecreasing and continuous. From (3), the number of jumps of the process under consideration on the interval $[t_1, t_2]$ into the set $A \subset G/e_1$ is distributed according to the Poisson law. The parameter of this Poisson distribution is equal to

$$\sum_{i, e_i \in A} m_i(t_2) - m_i(t_1).$$

Let us integrate the function y_{n-1}^{-1} along the curve $y(t)$. We have

$$\begin{aligned} L_j \left(\int_{e_1}^{y(t)} y^{-1} dy \right) &= L_j \left(\lim_{\Delta_n \rightarrow 0} \sum_{i=1}^{n-1} y^{-1}(t_i^{(n)}) (y(t_{i+1}^{(n)}) - y(t_i^{(n)})) \right) = \\ &= L_j \left(\lim_{\Delta_n \rightarrow 0} \sum_{i=0}^{n-1} y^{-1}(t_i^{(n)}) y(t_i^{(n)}) (y(t_i^{(n)}, t_{i+1}^{(n)}) - 1) \right) = \sum_{i, e_i \in T_j} m_i(t). \end{aligned}$$

By Theorem 3 this integral does not depend on the path of integration. Therefore the quantity

$$\sum_{i, e_i \in T_j} m_i(t)$$

will be the same for different curves y_1, y_2 , provided $y_1(t) = y_2(t)$.

We shall say that a set $A \subset G \setminus e_1$ is invariant if the distribution of jumps into this set on any interval $[t_1, t_2]$ depends only on the distribution of the increment of the process, i.e., on $y(t_1, t_2)$. Then, on the basis of what has been said, the following holds.

Theorem 4. *A set $A \subseteq G \setminus e_1$ is invariant if and only if it is the union of some number of conjugacy classes T_j .*

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REFERENCES

- ¹ S. Stoilov, *Theory of Functions of a Complex Variable*, 1, IL, 1962.
- ² V. M. Maksimov, *Probability Theory and Its Applications*, 12, no. 4 (1967).
- ³ V. M. Maksimov, DAN, 182, no. 1 (1968).

Note: Figure translations are in progress. See original paper for figures.

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