

SOME SOLUTIONS OF AN INHOMOGENEOUS DIFFUSION EQUATION

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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HYDROMECHANICS

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SOME SOLUTIONS OF AN INHOMOGENEOUS DIFFUSION EQUATION

(Presented by Academician L. I. Sedov on 11 X 1968)

We consider the following problem, close to those studied in works (¹⁻⁹): to find a bounded solution of the inhomogeneous diffusion equation

$$\partial u / \partial t = a^2 \partial^2 u / \partial x^2 + F[u(\bar{x}, t)], \quad (1)$$

periodic, or such that the initial function $u(x, 0)$ over a previously unknown time interval T increases by a factor k ($k > 1$), with the boundary condition

$$u(0, t) = 0 \quad (2)$$

in the semi-infinite domain $0 < x < \infty$. Here the function $F(u)$ has the form shown in Fig. 1, i.e. $F[u(\bar{x}, t)] = c$ ($c > 0$), if $u(\bar{x}, t) < u_*$, until $u(\bar{x}, t) = u_*$; if at $t = T_1$, $u(\bar{x}, T_1) = u_*$, then for $t > T_1$, $F[u(\bar{x}, t)] = -d$ ($d > 0$), if $u(\bar{x}, t) > u_{**}$ ($u_{**} < u_*$), until $u(\bar{x}, t) = u_{**}$ at $t = T$. In doing this we shall assume that

Fig. 1

$$u(x, T) = k\omega(x) \quad (\omega(x) = u(x, 0); \quad k \geq 1), \quad (3)$$

so that it must be

$$u_* = ku(\bar{x}, 0).$$

Thus, the formulation of the problem is analogous to that given in works (^{8, 9}).

Below, for $k \geq 1$, a solution of this problem is found for large values of T/T_1 and small d/c , under the assumption that the quantities c, d, T_1 , and T are related

by the relation $(c + d)T_1 = dT$. A solution of this problem is also obtained under the condition $\bar{x} \gg 2a\sqrt{T}$, and, if $k = 1$, then $(c + d)T_1 = dT$.

As an example of the problem described by equation (1) with conditions (2), (3), one may consider the following problem of filtration theory in a hydraulic formulation: let groundwater be situated in the region $0 \leq x < \infty$; let $u(x, t)$ denote the groundwater level; at the section $x = 0$ there is a wall of a canal in which a constant, in particular zero, liquid level is maintained; at the section $x = \bar{x}$ ($0 < \bar{x} < \infty$) the height of the groundwater mound is measured; as soon as the height of the mound reaches the value u_* , uniform irrigation carried out with infiltration intensity (taking evaporation into account) mc is stopped, and only evaporation with intensity md occurs; when the height of the mound has decreased to the value u_{**} , irrigation is again begun with the previous intensity; find the solution of the problem on the spreading of the groundwater mound. Here m is the active porosity of the soil, $a^2 = k\bar{u}/m$, k is the filtration coefficient, and $\bar{u}$ is the mean depth of the flow.

Let, for $0 < t < T_1$, $F[u(\bar{x}, t)] = c$, and for $T_1 < t < T$, $F[u(\bar{x}, t)] = -d$; at $t = T$ relation (3) is satisfied. We shall find the solution $u(x, t)$ of equation (1) with boundary condition (2).

The function $u(x, t)$, expressed in terms of the as yet unknown function $\omega(x)$, is determined by the formulas

$$\begin{aligned} u(x, t) &= v(c, x, t) + w(x, t) & (0 \leq t \leq T_1), \\ u(x, t) &= v(c, x, t) - v(c, x, t - T_1) - v(d, x, t - T_1) + w(x, t) & (T_1 \leq t \leq T), \end{aligned} \quad (4)$$

where

$$\begin{aligned} v(c, x, t) &= c \left[-\frac{x^2}{2a^2} + \left(t + \frac{x^2}{2a^2} \right) \Phi \left(\frac{x}{2a\sqrt{t}} \right) + \frac{x\sqrt{t}}{a\sqrt{\pi}} \exp \left(-\frac{x^2}{4a^2t} \right) \right], \\ w(x, t) &= \frac{\exp(-x^2/4a^2t)}{a\sqrt{\pi t}} \int_0^\infty \exp \left(-\frac{\xi^2}{4a^2t} \right) \operatorname{sh} \frac{x\xi}{2a^2t} \omega(\xi) d\xi, \\ \Phi(y) &= \frac{2}{\sqrt{\pi}} \int_0^y \exp(-S^2) dS. \end{aligned} \quad (5)$$

Condition (3) gives the integral equation

$$\varphi(x) = \psi(x) + \lambda \int_0^\infty K(x, \eta) \varphi(\eta) d\eta \quad (6)$$

for finding the function $\varphi(x)$. Here it is assumed that

$$\varphi(x) = \omega(x), \quad \psi(x) = f(x) \quad (k = 1),$$

$$\varphi(x) = -g(x) + \omega(x), \quad g(x) = D \left[1 + \sum_{m=1}^{\infty} \frac{1}{k^m} \Phi \left(\frac{x}{2a\sqrt{mT}} \right) \right], \quad (7)$$

$$\psi(x) = -D + f(x)/k \quad (k > 1),$$

where

$$f(x) = \gamma x^2 [1 - \Phi(\alpha x)] - \zeta x^2 [1 - \Phi(\beta x)] + \mu \Phi(\alpha x) + \chi \Phi(\beta x) - \delta x e^{-\alpha^2 x^2} + \varepsilon x e^{-\beta^2 x^2}, \quad (8)$$

$$\left(\gamma = \frac{c+d}{2a^2}, \quad \zeta = \frac{c}{2a^2}, \quad \alpha = \frac{1}{2a\sqrt{T-T_1}}, \quad \beta = \frac{1}{2a\sqrt{T}}, \right.$$

$$\mu = (c+d)(T_1 - T), \quad \chi = cT, \quad \delta = \frac{(c+d)\sqrt{T-T_1}}{a\sqrt{\pi}},$$

$$\varepsilon = \frac{c\sqrt{T}}{a\sqrt{\pi}},$$

$$D = [cT_1 + d(T_1 - T)]/k, \quad f(\infty) = Dk;$$

$$\lambda = \frac{1}{ka\sqrt{\pi T}}, \quad K(x, \eta, T) = \exp \left[-\frac{x^2 + \eta^2}{4a^2 T} \right] \operatorname{sh} \frac{x\eta}{2a^2 T}. \quad (9)$$

Consider the homogeneous integral equation corresponding to (6) for the case $k = 1$

$$\varphi(x) = \frac{1}{a\sqrt{\pi T}} \int_0^\infty \exp \left[-\frac{x^2 + \eta^2}{4a^2 T} \right] \operatorname{sh} \frac{x\eta}{2a^2 T} \varphi(\eta) d\eta. \quad (10)$$

As is known, for (10) the relation

$$\int_0^\infty \eta \exp \left(-\frac{\eta^2}{4a^2 T} \right) \operatorname{sh} \frac{x\eta}{2a^2 T} d\eta = a\sqrt{\pi T} x \exp \frac{x^2}{4a^2 T} \quad (11)$$

holds.

In consequence of (11), it is clear that the integral equation (10), for any B , has the solution $\varphi(x) = Bx$. Formulas (4), (5), and (7) show that, for boundedness of the solution $u(x, t)$ of the problem under consideration, the condition $B = 0$ must be satisfied.

Introducing the notation

$$R(x, \eta, T) = \sum_{m=1}^{\infty} \frac{1}{\sqrt{mk^{m-1}}} \exp \left[-\frac{x^2 + \eta^2}{4a^2mT} \right] \operatorname{sh} \frac{x\eta}{2a^2mT}, \quad (12)$$

one can see that, if in formula (8) $\psi(x) \rightarrow 0$ as $x \rightarrow \infty$, the solution $\varphi(x)$ of integral equation (6) is represented in the form

$$\varphi(x) = \psi(x) + \frac{1}{ka\sqrt{\pi T}} \int_0^{\infty} R(x, \eta, T) \psi(\eta) d\eta. \quad (13)$$

It follows from formulas (7) and (8) that for $k = 1$, $\psi(\infty) = D$, and for $k \neq 1$, $\psi(\infty) = 0$. For large m the general term of the series (12) has the form

$$\frac{1}{2a^2T} \frac{x\eta\psi(\eta)}{k^{m-1}m^{3/2}} + \dots$$

In the case $k = 1$, solution (13), by virtue of (8), exists only when $D = 0$, which gives the condition

$$cT_1 = d(T - T_1). \quad (14)$$

Thus, we have obtained the following result: the problem sought has a solution defined by formula (13), for $k > 1$, for arbitrary values of the constants c, d, T_1 , and T ; for $k = 1$, the solution of the problem exists only for certain values of c, d, T_1 , and T , namely those connected by equation (14).

The graphs of the function $f(x)$ for various values of the parameters c, d, T_1 , and T , connected by condition (14), are presented in Fig. 2. Here the solid curve corresponds to the values $c = 1$, $d = 1/99$, $T_1 = 1$, and $T = 100$, and the dashed curve to the values $c = 1$, $d = 1$, $T_1 = 1$, $T = 2$ ($a = 1$).

The constants T_1 and T are connected with the quantities u_* and u_{**} by the relations

$$u(\bar{x}, T_1) = u_*, \quad ku(\bar{x}, 0) = u_{**}. \quad (15)$$

In the case $k = 1$, the period of self-oscillation T is determined from equation (14), which must therefore be consistent with the equation $ku(\bar{x}, 0) = u_{**}$.

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Fig. 3

We now let, in formulas (4) and (5), the quantity T/T_1 tend to infinity and d/c tend to zero in such a way that condition (14) is preserved. From (8), (12), and (13) it is clear that $f(x) = \varphi(x) = \omega(x) = \psi(x) = 0$, and formulas (4) and (5) pass into the following:

$$\begin{aligned} u(x, t) &= v(c, x, t) & (0 \leq t \leq T_1), \\ u(x, t) &= v(c, x, t) - v(c, x, t - T_1) & (T_1 \leq t \leq T). \end{aligned} \quad (16)$$

Relations (16) give a solution of problem (1)–(3) for which

$$\lim_{t \rightarrow \infty} u(x, t) = ku(x, 0) = k\omega(x) = 0.$$

Formulas (16) and (5) show that, for any $\bar{x}$, $\partial u(\bar{x}, t)/\partial t > 0$ for $0 < t < T_1$, and $\partial u(\bar{x}, t)/\partial t < 0$ for $T_1 < t < T$.

The graph of the function $u = u(\bar{x}, t)/c$, given by relations (16), for the case $\bar{x} = 31.622777$, $d = 0$, $T_1 = 400$, $T = \infty$, $a = 1$, is shown in Fig. 3.

Let now the quantity T/T_1 be large, and d/c small, with condition (14) satisfied. Expanding the functions $\psi(x)$ and $R(x, \eta, T)$, given by formulas (7), (8), and (12), in series in powers of $T^{-1/2}$ and T^{-1} , respectively, and substituting these series into equation (13), we find an asymptotic representation of the function $\varphi(x)$ for large values of T . The first term of this expansion has the form

$$\varphi(x) = \frac{\nu}{\sqrt{T}} x + \dots,$$

where

$$\nu = \frac{cT_1}{a\sqrt{\pi}}(-1 + S), \quad S = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \dots (2n+1) \sigma(n + 3/2)}{2^{n+1} n! (n+2)}.$$

Here $\sigma(s)$ denotes the series convergent for $k \geq 1$,

$$\sigma(s) = \sum_{m=1}^{\infty} \frac{1}{k^m m^s},$$

which for $k = 1$ becomes the Riemann zeta function $\zeta(s)$ ⁽¹¹⁾. It is easy to see that the series S entering the expression for the quantity ν also converges.

Relations (4) and (5) show that, in the case $\bar{x} \gg 2a\sqrt{T}$, and also for sufficiently large T/T_1 and sufficiently small d/c , if $\bar{x} \leq 2a\sqrt{T}$, the conditions $\partial u(\bar{x}, t)/\partial t > 0$ for $0 < t < T_1$, $\partial u(\bar{x}, t)/\partial t < 0$ for $T_1 < t < T$ are also satisfied, and the quantities T_1 and T can be found from relations (14) and (15).

Let us now consider problem (1)–(3) for $k = 1$ on the finite interval $0 < x < l$. The function $u(x, 0) = \varphi(x)$ is determined by formulas (12)–(13), where $R(x, \eta, T)$ is replaced by

$$R(x, \eta, T) + \sum_{s=1}^{\infty} \{R(x - 2sl, \eta, T) + R(x + 2sl, \eta, T)\},$$

and

$$\psi(x) = \sum_{n=-\infty}^{\infty} [2\Lambda(\alpha_n) - \Lambda(\beta_n) + \Lambda(\gamma_n)],$$

$$\Lambda(\alpha) = \alpha^2 \left[cM(\alpha/\sqrt{T}) - (c + d)M(\alpha/\sqrt{T - T_1}) \right],$$

$$M(y) = \exp(-y^2)/\sqrt{\pi}y + [1 + 1/2y^2]\Phi(y) - 1,$$

$$\alpha_n = (x - 2nl)/2a, \quad \beta_n = \alpha_n + l/2a, \quad \gamma_n = -\alpha_n + l/2a.$$

From the results obtained in work ⁽⁹⁾ there follows the existence of a solution of the problem under consideration for $k = 1$ in the finite domain $0 \leq x \leq l$.

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