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Abstract

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MATHEMATICS

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AN EXAMPLE OF A CONDENSING OPERATOR IN THE THEORY OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT OF NEUTRAL TYPE

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In the present article the question of the existence of solutions of an equation of neutral type is investigated

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))]. \quad (1)$$

If the function $f(t, x, y, z)$ satisfies the Lipschitz condition with respect to the variables x, y, z , with constants k_x, k_y, k_z , respectively, and moreover $k_z < 1$, then under small additional assumptions the question of the existence of solutions is easily reduced to the contraction mapping principle (see ⁽¹⁾). Here we shall abandon the Lipschitz condition with respect to the variables x, y . To prove the existence theorem in this case, the fixed-point principle for condensing operators ⁽²⁾ will be applied. As P. P. Zabreiko observed, this problem can be reduced in another way to Schauder's principle.

1. Let us recall the fixed-point principle from ⁽²⁾. Let E be a Banach space and $\Omega \subseteq E$. The numerical set $Q(\Omega)$ is defined as follows: a positive number ε belongs to $Q(\Omega)$ if and only if the set Ω has a finite ε -net. The measure of noncompactness of the set Ω is called the quantity

$$\chi(\Omega) = \inf Q(\Omega).$$

(See also ⁽³⁻⁵⁾.) It is clear that if in the definition of the set $Q(\Omega)$ the finite ε -net is replaced by a compact one, then the quantity $\chi(\Omega)$ will not change. Let $T \subseteq E$. An operator $A : T \rightarrow E$ is called condensing if it is continuous on T and, for any noncompact set Ω , the inequality

$$\chi[A(\Omega)] < \chi(\Omega)$$

holds.

Theorem 1. *If a condensing operator A maps a convex closed bounded set T into itself, then it has at least one fixed point in T .*

2. We shall consider (1) together with the initial condition

$$x(t) = x_0(t) \quad (-h < t \leq 0), \quad (1')$$

where $x_0(t)$ is a fixed function defined on the half-interval (finite or infinite) $(-h, 0]$. By a solution of problem (1)–(1') we shall understand a function $x(t)$ $(-h < t \leq H]$, satisfying the initial condition (1') and the following three requirements: a) $x(t)$ is continuous on $(-h, H]$; b) $x'(t)$ exists almost everywhere on $(-h, H]$ and is summable to the power $p \geq 1$; c) almost everywhere on $[0, H]$

$$x'(t) = f[t, x(t), x(t - h_1(t)), x'(t - h_2(t))].$$

We shall denote by $\mathcal{E}(0, H)$ the set of functions continuous on $[0, H]$ and possessing a derivative summable to the power p ; this set becomes a Banach space with the natural linear operations if one sets $\|x\|_{\mathcal{E}} = \|x\|_C + \|x'\|_{\mathcal{L}_p}^*$ ⁽¹⁾. For any function

* $\mathcal{E}(0, H)$ is the Sobolev space W_p^1 up to an equivalent norm.

for $x(t) \in \mathcal{E}(0, H)$ we put

$$\tilde{x}(t) = \begin{cases} x_0(t), & -h < t < 0, \\ x(t), & 0 \leq t \leq H. \end{cases}$$

Along with problem (1)–(1'), let us consider the operator equation in the space $\mathcal{E}(0, H)$:

$$y = Iy, \quad (2)$$

where the operator I is defined by the formula

$$Iy(t) = x_0 + \int_0^t f[s, y(s), \tilde{y}(s - h_1(s)), \tilde{y}'(s - h_2(s))] ds$$

$$(x_0 = x_0(0)).$$

It is not hard to verify that if the function $x_0(t)$ is continuous, and its derivative is summable with power p , then equation (2) is equivalent to problem (1)–(1')

in the following sense: if $x(t)$ is a solution of problem (1)—(1'), then its restriction $y(t)$ to the interval $[0, H]$ is a solution of equation (2), and conversely, if $y(t)$ is a solution of equation (2), then the function $x(t) = \tilde{y}(t)$ is a solution of problem (1)—(1').

3. Let us establish some properties of the operator I .

Lemma 1. Let $\mathcal{E}_0$ be the set of functions from $\mathcal{E}(0, H)$ satisfying the condition $x(0) = x_0$. Suppose that the functions $x_0(t)$, $h_1(t)$, $h_2(t)$, and $f(t, x, y, z)$ satisfy the following requirements: (I) $x_0(t)$ is continuous and bounded, and $x'_0(t)$ is summable with power p on $(-h, 0]$; (II) $-H + t \leq h_i(t) \leq h + t$ ($i = 1, 2$; $0 \leq t \leq H$); (III) $h_1(t)$, $h_2(t)$ are measurable on $[0, H]$; (IV) the function $q(t) = t - h_2(t)$ is such that: a) the inverse image of every set of measure zero is measurable and b) for any measurable set $E \subseteq [0, H]$ satisfying the condition $q(E) \subseteq [0, H]$, the inequality $\mu E \leq r\mu q(E)$ holds (the number r does not depend on E); (V) $f(t, x, y, z)$ is defined for $0 \leq t \leq H$ and arbitrary real x, y, z ; (VI) $f(t, x, y, z)$ is measurable in t for any fixed x, y, z ; (VII) $f(t, x, y, z)$ is continuous jointly in x, y for fixed t, z ; (VIII) $f(t, x, y, z)$ satisfies a Lipschitz condition in z :

$$|f(t, x, y, z_1) - f(t, x, y, z_2)| \leq k|z_1 - z_2|;$$

(IX) for any $R > 0$ one can specify a function $m_R(t) \in \mathcal{L}_p(0, H)$ such that

$$|f(t, x, y, z)| \leq m_R(t)$$

$$(0 \leq t \leq H; |x - x_0|, |y - x_0| \leq R; -\infty < z < \infty).$$

Then the operator I maps $\mathcal{E}_0$ into $\mathcal{E}_0$ and is continuous.

Proof. It is not hard to show that the operator I maps $\mathcal{E}_0$ into $\mathcal{E}_0$. We shall show that it is continuous.

Let $y_n, y \in \mathcal{E}_0$ and $y_n \rightarrow y$, i.e. $\|y_n - y\|_C \rightarrow 0$, $\|y'_n - y'\|_{\mathcal{L}_p} \rightarrow 0$. Let $z_n = Iy_n$, $z = Iy$. Put

$$u_n(t) = f[t, y_n(t), \tilde{y}_n(t - h_1(t)), \tilde{y}'(t - h_2(t))]$$

(this function, as is not hard to see, also belongs to $\mathcal{L}_p(0, H)$). Then

$$\|z'_n - z'\|_{\mathcal{L}_p} \leq \|z'_n - u_n\|_{\mathcal{L}_p} + \|u_n - z'\|_{\mathcal{L}_p}. \quad (3)$$

The second summand tends to zero by conditions (VII), (IX), and Lebesgue's theorem on passage to the limit under the integral sign. We estimate the first summand with the aid of the Lipschitz condition:

$$\|z'_n - u_n\|_{\mathcal{L}_p} \leq k \left\{ \int_0^H |\tilde{y}'_n(s - h_2(s)) - \tilde{y}'(s - h_2(s))|^p ds \right\}^{1/p}. \quad (4)$$

Put $E = \{s \mid 0 \leq s \leq H, 0 \leq q(s) \leq H\}$. Obviously,

$$\int_0^H |\tilde{y}'_n(s - h_2(s)) - \tilde{y}'(s - h_2(s))|^p ds \leq \int_E |y'_n(s - h_2(s)) - y'(s - h_2(s))|^p ds, \quad (5)$$

since for $s \notin E$

$$\tilde{y}'_n(s - h_2(s)) = \tilde{y}'(s - h_2(s)) = x'_0(s - h_2(s)).$$

Further, from condition (IV) there follows the inequality

$$\int_E |y'_n(s - h_2(s)) - y'(s - h_2(s))|^p ds \leq r \int_{q(E)} |y'_n(s) - y'(s)|^p ds. \quad (6)$$

To prove this fact, introduce the notation $\psi(t) = |y'_n(t) - y'(t)|^p$ and consider the set

$$E_i = \{s \in E \mid a_{i-1} \leq \psi(s - h_2(s)) < a_i\},$$

where a_{i-1}, a_i are constants. Obviously,

$$q(E_i) = \{s \in q(E) \mid a_{i-1} \leq \psi(s) < a_i\}.$$

Taking condition (IV) into account, we obtain the following relation for the integral sums of the integrals occurring in (6): $\sigma_1 \leq r\sigma_2$. Hence inequality (6) is obtained by passage to the limit. From (4), (5), and (6) we obtain:

$$\|z'_n - u_n\|_{\mathcal{L}_p} \leq kr^{1/p} \|y'_n - y'\|_{\mathcal{L}_p}.$$

Consequently, the first term in (3) also tends to zero. Thus,

$$\|z_n - z'\|_{\mathcal{L}_p} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (7)$$

We now estimate the quantity $\|z_n - z\|_C$:

$$\|z_n - z\|_C = \max_{0 \leq t \leq H} |z_n(t) - z(t)| \leq$$

$$\leq \int_0^H |z'_n(s) - z'(s)| ds \leq H^{1/q} \|z'_n - z'\|_{\mathcal{L}_p} \left(\frac{1}{q} + \frac{1}{p} = 1 \right).$$

Hence, also from (7), it follows that $\|z_n - z\|_C \rightarrow 0$ as $n \rightarrow \infty$. The continuity of the operator I is proved.

Remark. Consider the operator $F_y(x)$, defined by the formula

$$F_y x(t) = f[t, x(t), \tilde{x}(t - h_1(t)), \tilde{y}(t - h_2(t))]$$

$$(y \in \mathcal{L}_p(0, H), x \in C(0, H), \tilde{y}(t) = x'_0(t) \text{ for } t < 0).$$

Arguing as above, one can verify that, under the conditions of Lemma 1, this operator maps $C(0, H)$ into $\mathcal{L}_p(0, H)$ and is continuous.

Our next task is the construction of the set T occurring in Theorem 1. Fix $R > 0$ (so that for $-h < t \leq 0$ the inequality $|x_0(t) - x_0| \leq R$ holds) and denote by T_1 the set of functions from $\mathcal{E}_0$ satisfying the inequality $\|x - x_0\|_{\mathcal{E}} \leq R$. Choose $H > 0$ so small that the inequality

$$\int_0^H m_R(t) dt + \|m_R(t)\|_{\mathcal{L}_p} \leq R$$

holds.

Then $IT_1 \subseteq T_1$. Indeed,

$$\begin{aligned} \|Ix - x_0\|_{\mathcal{E}} &= \|Ix - x_0\|_C + \|(Ix)'\|_{\mathcal{L}_p} \leq \\ &\leq \int_0^H |f[s, x(s), \tilde{x}(s - h_1(s)), \tilde{x}'(s - h_2(s))]| ds + \|m_R(t)\|_{\mathcal{L}_p} \leq R. \end{aligned}$$

The set T_1 is convex, closed, and bounded; however, the operator I on this set is not, generally speaking, condensing. Put $T = \overline{\text{co}} IT_1$. It is not hard to see that the set T is also convex, closed, and bounded, and moreover $IT \subseteq T$.

Lemma 2. Suppose the conditions of Lemma 1 are satisfied. Suppose, in addition, that the condition

$$(X) \quad kr^{1/p} < \begin{cases} 1, & \text{if } p > 1, \\ 1/2, & \text{if } p = 1. \end{cases}$$

Then the operator I is condensing on T , if H is sufficiently small.

Proof. If $\Omega \subseteq T$, then by the symbol $(\Omega)'$ we shall denote the set of derivatives of functions from the set Ω . The assertion of the lemma follows from the inequalities

$$\chi_{\mathcal{E}}(I\Omega) \leq (1 + H^{1/q})\chi_{\mathcal{L}_p}[(I\Omega)'], \quad (8)$$

$$\chi_{\mathcal{L}_p}[(I\Omega)'] \leq kr^{1/p}\chi_{\mathcal{L}_p}[(\Omega)']; \quad (9)$$

$$\chi_{\mathcal{L}_p}[(\Omega)'] \leq \chi_{\mathcal{E}}(\Omega) \quad (10)$$

and condition (X).

We shall not dwell on the proof of inequalities (8), (10). Let us verify the validity of inequality (9). First note that the set Ω is compact in the space $C(0, H)$, since it is uniformly bounded and equicontinuous. Let $\{y_i; i = 1, 2, \dots, n\}$ be an ε -net of the set $(\Omega)'$ in $\mathcal{L}_p(0, H)$. Construct the sets $S_i = F_{y_i}(\Omega)$ (see the remark to Lemma 1). The sets S_i are compact in $\mathcal{L}_p(0, H)$, since the operators F_{y_i} are continuous. Consequently, the union of these sets

$$S = \bigcup_{i=1}^n S_i$$

is also compact. We show that S is a $kr^{1/p}\varepsilon$ -net of the set $(I\Omega)'$ in $\mathcal{L}_p(0, H)$. Let $z \in (I\Omega)'$, i.e.

$$z(t) = f[t, y(t), \tilde{y}(t - h_1(t)), \tilde{y}'(t - h_2(t))],$$

where $y \in \Omega$. Let $\|y' - y_{i_0}\|_{\mathcal{L}_p} \leq \varepsilon$. Choose in the set S_{i_0} the element $u = F_{y_{i_0}}(y)$. Then

$$\|z - u\|_{\mathcal{L}_p} \leq k \left\{ \int_0^H |\tilde{y}'(s - h_2(s)) - \tilde{y}_{i_0}(s - h_2(s))|^p ds \right\}^{1/p}.$$

Arguing as in the proof of Lemma 1, we obtain

$$\left\{ \int_0^H |\tilde{y}'(s - h_2(s)) - \tilde{y}_{i_0}(s - h_2(s))|^p ds \right\}^{1/p} \leq r^{1/p} \|y' - y_{i_0}\|_{\mathcal{L}_p}.$$

Thus,

$$\|z - u\|_{\mathcal{L}_p} \leq kr^{1/p}\varepsilon,$$

i.e. from a finite ε -net of the set $(\Omega)'$ we can construct a compact $kr^{1/p}\varepsilon$ -net of the set $(I\Omega)'$. This means precisely that inequality (9) holds. From (8)–(10) we obtain

$$\chi_{\mathcal{E}}(I\Omega) \leq \Theta \chi_{\mathcal{E}}(\Omega),$$

where $\Theta = (1 + H^{1/q})kr^{1/p}$.

It is obvious that for sufficiently small H , condition (X) implies the inequality $\Theta < 1$ (if $p = 1$, then $q = \infty$ and $H^{1/q} = 1$). The lemma is proved.

From Lemmas 1, 2 and the fixed-point principle for condensing operators we obtain a theorem on the solvability of problem (1)–(1').

Theorem 2. Let the functions $x_0(t), h_1(t), h_2(t)$ and $f(t, x, y, z)$ satisfy conditions (I)–(X).

Then problem (1)–(1') has a solution $x(t)$, defined on some semi-interval $(-h, H]$ ($H > 0$).

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