

ON AN EXAMPLE OF A GRAPH HAVING NO TRANSITIVE AUTOMORPHISM GROUP

G. M. Adel'son-Vel'skii, B. Yu. Weisfeiler, A. A. Leman, I. A. Faradzhhev

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.45654>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.95

MATHEMATICS

G. M. ADELSON-VELSKII, B. Yu. WEISFEILER,
A. A. LEMAN, I. A. FARADZHEV

ON AN EXAMPLE OF A GRAPH HAVING NO TRANSITIVE AUTOMORPHISM GROUP

(Presented by Academician P. S. Novikov on 29 VII 1968)

1. In paper ⁽¹⁾ a construction was indicated which makes it possible, for a given graph Γ , to construct in an invariant way the matrix algebra $\mathfrak{A}(\Gamma)$. In the present paper we consider graphs to which there corresponds a special class of algebras—cells.

An a priori definition of a cell:

A matrix algebra $\mathfrak{A}$ is called a **cell** if it has the following properties:

1. $\mathfrak{A}$ is invariant under transposition.
2. In $\mathfrak{A}$ there exists a basis $e_0 = E, e_1, \dots, e_k$ such that e_i is a matrix consisting of zeros and ones and having n_i ones in each row and each column,

$$n = \sum_{i=1}^k n_i + 1; \quad \sum_{i=1}^k e_i$$

is a matrix all of whose entries are equal to 1.

Let Γ be a graph and $\mathfrak{A}(\Gamma)$ its algebra. The condition that $\mathfrak{A}(\Gamma)$ is a cell has the following geometric meaning. To each ordered sequence $(g_0, g_1, \dots, g_m)$ of vertices of the graph Γ ($m \geq 1$) there corresponds the sequence $(\xi_1, \dots, \xi_m)$, where $\xi_i = 1$ if (g_{i-1}, g_i) is an edge of Γ , and $\xi_i = 0$ otherwise. If $\mathfrak{A}(\Gamma)$ is a cell, then for any fixed sequence $\Xi_0 = (\xi_1, \dots, \xi_k)$ of zeros and ones the number of sequences of vertices $(g \dots)$ and $(h \dots)$ to which the sequence Ξ_0 corresponds is the same for any two vertices g and h .

II. In ⁽¹⁾ the following was stated.

Hypothesis. The automorphism group $\text{Aut } \Gamma$ of a graph Γ , the algebra $\mathfrak{A}(\Gamma)$ of which is a cell, is transitive on its vertices.

Below a counterexample to this hypothesis is constructed.

A. Let Γ be a graph, $G = \text{Aut } \Gamma$, and $\mathfrak{Z}(G)$ the algebra of matrices commuting with the group G . $\mathfrak{Z}(G)$ has a basis f_i , $i \in \mathfrak{I}$, all of whose matrices consist of 0 and 1. It is known that $\mathfrak{A}(\Gamma)$ is a subalgebra of $\mathfrak{Z}(G)$, and

$$e_i = \sum_{j \in \mathfrak{I}_i} f_j,$$

where the $\mathfrak{I}_i$ are suitable disjoint subsets of the set $\mathfrak{I}$.

B. If $n = 2p$, p is prime, and G is a primitive group, transitive but not doubly transitive on the vertices of the graph Γ (with n vertices), then, as shown in ⁽²⁾, $\dim \mathfrak{Z}(G) = 3$, $2p = m^2 + 1$, $n_1 = m(m+1)/2$, $n_2 = m(m-1)/2$. Hence, and from what was said above, it follows that $\dim \mathfrak{A}(\Gamma) \leq 3$; the case $\dim \mathfrak{A}(\Gamma) = 2$ is trivial and corresponds to the complete graph Γ . Therefore we shall assume that $\mathfrak{A}(\Gamma) = \mathfrak{Z}(G)$. Note that the matrices e_i in this case are symmetric and are, therefore, adjacency matrices of undirected graphs without multiple edges. Moreover, $\mathfrak{A}(e_i) = \mathfrak{Z}(G)$, $i > 0$.

As was indicated in ⁽²⁾, and also in a later paper ⁽³⁾, it is unknown whether there exists a primitive not doubly transitive group G for $p > 5$.

C. For $p = 13$ all three-dimensional cells were constructed whose automorphism group contains the element $\sigma = (1, 2, \dots, p)(p+1, \dots, 2p)$.

It turned out that the automorphism groups of all the corresponding graphs are intransitive. Hence, and from the results of [2], it follows that

Every primitive transitive permutation group of degree 26 is doubly transitive.

III. In view of IIA, the basic matrix of the e_1 -cell, permutable with σ , has the form

$$\begin{pmatrix} A_1 A_2 \\ A_3 A_4 \end{pmatrix},$$

where $A'_3 = A_2$, $A'_1 = A_1$, $A'_4 = A_4$, $A_i = \sum_{j \in \mathfrak{S}_i} \tau^j$, where τ is the permutation matrix of $(0, 1, \dots, p-1)$, and $\mathfrak{S}_i$ are suitable subsets of residues mod p ; $\mathfrak{S}_1$ and $\mathfrak{S}_4$ do not contain 0.

The choice of the suitable sets $\mathfrak{S}_i$ was carried out by direct search with the aid of a computer. In all, 104 different collections of $\mathfrak{S}_i$ were obtained. In all cases either

$$\mathfrak{S}_1 = \{1, 3, 4, 9, 10, 12\}, \quad \mathfrak{S}_4 = \{2, 5, 6, 7, 8, 11\},$$

or conversely; for fixed $\mathfrak{S}_2$ these two cases are, evidently, isomorphic. For given $\mathfrak{S}_1$ and $\mathfrak{S}_4$, 52 collections of $\mathfrak{S}_2$ were obtained; the corresponding graphs are pairwise isomorphic. The isomorphisms consist of permutations belonging to the group generated by the substitutions

$$\begin{pmatrix} E & 0 \\ 0 & \tau \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \rho \\ \rho^{-1} & 0 \end{pmatrix}$$

for

$$\rho = (0)(1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7);$$

one of the admissible collections is

$$\mathfrak{S}_2 = \{0, 1, 3, 9\}.$$

The intransitivity of the obtained graph Γ is established by comparing the subgraphs Γ_1 and Γ_2 , spanned by the vertices adjacent respectively to the 1st and 14th vertices of the graph Γ (Fig. 1).

Fig. 1

Received
5 VI 1968

REFERENCES

1. B. Yu. Weisfeiler, L. L. Leman, *Scientific and Technical Information*, No. 9, issue 2 (1968).
2. H. Wielandt, *Math. Zs.*, **63**, 478 (1955).
3. D. G. Higman, *Math. Zs.*, **86**, No. 2, 145 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.