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V. R. Portnov

1969

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Abstract

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UDC 513.88

MATHEMATICS

V. R. PORTNOV

ON A PROJECTION OPERATOR OF S. L. SOBOLEV TYPE

(Presented by Academician S. L. Sobolev on 9 IV 1969)

In the work of S. L. Sobolev (¹), for the space $L_p^{(m)}(\Omega)$ a projection operator Π was constructed whose range is the space of polynomials of degree not greater than $m - 1$. This operator Π has the property that for any function $u(x) \in L_p^{(m)}(\Omega)$ there is a representation

$$u(x) - \Pi u(x) = \sum_{|\alpha|=m} Q_\alpha(D^\alpha u(x)), \quad (1)$$

where Q_α is a certain linear bounded operator mapping $L_p(\Omega)$ into itself. Representations of the form (1) were later generalized in various directions (see (²⁻⁴)). In (⁵), applications of such representations were given to the proof of solvability of the first boundary-value problem for one class of degenerate equations.

In the present note a projection operator Π is constructed, and a representation of the form (1) is obtained for a certain collection $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_R$ of partial differential operators that are products of ordinary differential operators with respect to different variables. The corresponding result of (⁵) is contained here as a special case.

Let us first introduce a number of notations: n and R are a fixed pair of natural numbers; the letters i and k below will denote natural numbers not exceeding n and R , respectively; $\bar{E}$ is the extended real line; E_n is the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$. The remaining notations will be introduced in the course of the exposition.

Suppose that for each i an interval (a_i, c_i) is given on the axis Ox_i of the space E_n , where a_i and $c_i \in \bar{E}$. This means that $-\infty \leq a_i < c_i \leq \infty$. Put $D = (a_1, c_1) \times \dots \times (a_n, c_n)$.

Let, further, to each k there correspond a set $(m_{1,k}, m_{2,k}, \dots, m_{n,k})$ of n non-negative integers and an operator $\mathcal{L}_k = \mathcal{L}_{1,k} \mathcal{L}_{2,k} \dots \mathcal{L}_{n,k}$, whose domain is the space $C^{(\infty)}(D)$, and for each i

$$\mathcal{L}_{i,k} = \begin{cases} \frac{\partial^{m_{i,k}}}{\partial x_i^{m_{i,k}}} + \sum_{j=1}^{m_{i,k}} b_{i,k}^{(j)}(x_i) \frac{\partial^{j-1}}{\partial x_i^{j-1}}, & \text{if } m_{i,k} > 0, \\ I \text{ (} I \text{ is the identity operator),} & \text{if } m_{i,k} = 0, \end{cases}$$

where $b_{i,k}^{(j)}(x_i) \in L_1^{\text{loc}}(D)$ for all $j = 1, \dots, m_{i,k}$.

If $m_{i,k} > 0$, then the operator $\mathcal{L}_{i,k}$ may be regarded as an ordinary differential operator on the interval (a_i, c_i) . Let $\{\varepsilon_{i,k}^{(j)}(x_i)\}$ ($j = 1, \dots, m_{i,k}$) be its fundamental system of solutions on the indicated interval. We denote the linear span of this system by $S_{i,k}$. If $m_{i,k} = 0$, then put $S_{i,k} = \{0\}$, where $\{0\}$ denotes the set consisting of the single function identically equal to zero on (a_i, c_i) . This notation will be used below.

For each pair (i, k) let us consider the set $\Psi_{i,k}$ of all possible tuples $\tau = (k_1, \dots, k_l)$ ($1 \leq l \leq R$ and l , generally speaking, depends on τ) of natural numbers satisfying the following conditions: 1) $k < k_1 < \dots < k_l \leq R$; 2) there is a tuple $(i_1, i_2, \dots, i_R)$ of R natural numbers not exceeding n such that $m_{i_1,1} > 0, \dots, m_{i_R,R} > 0$, and, moreover,

$$i_k = i_{k_1} = \dots = i_{k_l} = i,$$

while $i_j \neq i$ for $j > k$ and $j \neq k_1, \dots, j \neq k_l$.

For some pairs (i, k) it may happen that $\Psi_{i,k} = \emptyset$. For example, $\Psi_{i,R} = \emptyset$ for all i , and $\Psi_{i,k} = \emptyset$ when $m_{i,k} = 0$.

If $\Psi_{i,k} \neq \emptyset$, then for any tuple $\tau = (k_1, \dots, k_l) \in \Psi_{i,k}$ we introduce the sets $S_{i,k}^{(\tau)}$ and $\Gamma_{i,k}^{(\tau)}$ of functions defined on the interval (a_i, c_i) :

$$S_{i,k}^{(\tau)} = \bigcap_{1 \leq j \leq l} S_{i,k_j}; \quad \Gamma_{i,k}^{(\tau)} = S_{i,k} \cap S_{i,k}^{(\tau)}.$$

Now, for each pair (i, k) , we formulate

Condition A. One of the following assertions is true:

- 1) $\Psi_{i,k} = \emptyset$;
- 2) $\Psi_{i,k} \neq \emptyset$, and for every $\tau \in \Psi_{i,k}$ there exists a subspace

$$\nabla_{i,k}^{(\tau)} \subset S_{i,k}^{(\tau)}$$

such that

$$S_{i,k}^{(\tau)} = \Gamma_{i,k}^{(\tau)} \oplus \nabla_{i,k}^{(\tau)},$$

and

$$S_{i,k} \cap \nabla_{i,k} = \{0\},$$

where $\nabla_{i,k}$ is the space spanned by

$$\bigcup_{\tau \in \Psi_{i,k}} \nabla_{i,k}^{(\tau)}.$$

(For the notation and terminology, see ⁽⁶⁾, pp. 30 and 44, 46.)

Everywhere in what follows it is assumed that for every pair (i, k) condition A is fulfilled and, in the case when $\Psi_{i,k} \neq \emptyset$, the family of spaces

$$\{\nabla_{i,k}^{(\tau)}\} \quad (\tau \in \Psi_{i,k})$$

is fixed.

Lemma. Let $m_{i,k} > 0$. Then there exists a system of functions

$$\{f_{i,k}^{(j)}(x_i)\} \quad (j = 1, \dots, m_{i,k}),$$

defined on the interval (a_i, c_i) , finite and infinitely differentiable, which has the property that

$$\int_{a_i}^{c_i} f_{i,k}^{(j)}(x_i) e_{i,k}^{(s)}(x_i) dx_i = \delta_{js},$$

where δ_{js} is the Kronecker symbol, and, in the case when $\Psi_{i,k} \neq \emptyset$,

$$\int_{a_i}^{c_i} f_{i,k}^{(j)}(x_i) v(x_i) dx_i = 0$$

for any function $v(x_i) \in \nabla_{i,k}$ and for all $j = 1, \dots, m_{i,k}$.

If $m_{i,k} > 0$, then the functions $\{e_{i,k}^{(j)}(x_i)\}$ and the functions $\{f_{i,k}^{(j)}(x_i)\}$ ($j = 1, \dots, m_{i,k}$) appearing in the lemma will be regarded as fixed and defined on the whole domain D .

We construct the projection operator $\Pi_{i,k}$, defining it on functions $u(x)$ from $L_1^{\text{loc}}(D)$, in the case $m_{i,k} > 0$, as follows:

$$\Pi_{i,k} u(x) = \sum_{j=1}^{m_{i,k}} \left(\int_{a_i}^{c_i} f_{i,k}^{(j)}(\xi_i) u(x_1, \dots, x_{i-1}, \xi_i, x_{i+1}, \dots, x_n) d\xi_i \right) e_{i,k}^{(j)}(x_i).$$

If $m_{i,k} = 0$, then $\Pi_{i,k}$ will be regarded as the zero operator on the space $L_1^{\text{loc}}(D)$.

We now define the operators Π_k and Π , mapping the space $L_1^{\text{loc}}(D)$ into itself. Put

$$\begin{aligned} \Pi_k &= I - (I - \Pi_{1,k})(I - \Pi_{2,k}) \cdots (I - \Pi_{n,k}) \quad (k = 1, \dots, R), \\ \Pi &= \Pi_1 \Pi_2 \cdots \Pi_R \quad (I \text{ is the identity operator}). \end{aligned}$$

Theorem 1. The following equalities hold:

$$\Pi_k^2 = \Pi_k, \quad \Pi_k \Pi = \Pi \quad (k = 1, \dots, R); \quad \Pi^2 = \Pi.$$

Definition. An additive and homogeneous operator Q mapping the space $L_1^{\text{loc}}(D)$ into itself will be called **strongly continuous** if for every compact set $K^{(1)} \subset D$ there exists a compact set $K^{(2)} \subset D$, depending, generally speaking, on $K^{(1)}$, such that for all functions $w(x) \in L_1^{\text{loc}}(D)$ the inequality

$$\|Qw(x)\|_{L_1(K^{(1)})} \leq C\|w(x)\|_{L_1(K^{(2)})},$$

holds, where C is a real number independent of $w(x)$.

Theorem 2. For any function $u(x) \in C^{(\infty)}(D)$ and any k , the equality

$$u(x) - \Pi_k u(x) = T_k(\mathcal{L}_k u(x)), \quad (2)$$

holds, where T_k is a strongly continuous operator in the space $L_1^{\text{loc}}(D)$, possessing the property that $T_k w(x) = 0$ implies $w(x) = 0$.

Theorem 3. For any function $u(x) \in C^{(\infty)}(D)$ and any collection $\beta = (\beta_1, \dots, \beta_n)$ of nonnegative integers such that $\beta_i \leq \min_{1 \leq k \leq R} m_{i,k}$ ($i = 1, \dots, n$), the equality

$$\mathcal{D}^\beta(u(x) - \Pi u(x)) = \sum_{k=1}^R Q_k^{(\beta)}(\mathcal{L}_k u(x)), \quad (3)$$

holds, where $Q_k^{(\beta)}$ ($k = 1, \dots, R$) are strongly continuous operators in the space $L_1^{\text{loc}}(D)$.

Remark 1. The operators $\Pi_{i,k}$, Π_k , and Π are, obviously, strongly continuous in the space $L_1^{\text{loc}}(D)$.

Remark 2. Equalities (2) and (3) are in fact valid, of course, for a class of functions broader than $C^{(\infty)}(D)$. This class will not be defined here; however, let us note that, for example, equality (3) with $\beta = (0, \dots, 0)$, and in the case when all the operators $\mathcal{L}_{i,k}$ with $m_{i,k} > 0$ have constant coefficients, is valid for all functions for which in the domain D there exist generalized, in the sense of S. L. Sobolev, operators $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_R$, as is easily verified by means of passage to the limit in equality (3).

Received
24 III 1969

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