

MAGNITUDES OF DEVIATIONS OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

V. P. Petrenko

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.44734>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.535.6

MATHEMATICS

V. P. PETRENKO

MAGNITUDES OF DEVIATIONS OF MEROMORPHIC FUNCTIONS OF LOWER ORDER LESS THAN ONE

(Presented by Academician M. A. Lavrent'ev on 29 XI 1968)

§ 1. Consider a function $f(z)$, meromorphic for $z \neq \infty$, of finite lower order λ and order ρ . In [1] the properties were considered of the quantities

$$\beta(a; f) = \lim_{r \rightarrow \infty} \frac{\ln^+ M(r, a, f)}{T(r, f)},$$

where a is an arbitrary complex number from the extended complex plane,

$$M(r, a, f) = \max_{|z|=r} \frac{1}{|f(z) - a|}, \quad a \neq \infty,$$

$$M(r, \infty, f) = \max_{|z|=r} |f(z)|,$$

and $T(r, f)$ is the Nevanlinna characteristic of $f(z)$.

It turned out (see [1]) that the quantities $\beta(a, f)$ for meromorphic functions of finite lower order possess a number of properties analogous to the properties of the deficiency quantities of R. Nevanlinna [2].

In particular, the following assertion is valid:

Theorem A [1]. *For an arbitrary meromorphic function $f(z)$ of finite lower order λ , the set*

$$\Omega(f) = \{a : \beta(a, f) > 0\}$$

is at most countable and

$$\sum_{(a)} \beta(a, f) \leq K_1(\lambda + 1),$$

where K_1 is an absolute constant.

Definition. We shall call the quantity $\beta(a, f)$ the **magnitude of deviation** of the meromorphic function $f(z)$ from the number a . If for some number a , $\beta(a, f) > 0$, then we shall say that $f(z)$ has a **positive deviation** from the number a .

It follows from Theorem A that the magnitudes of deviations of meromorphic functions of finite lower order are equal to zero for almost all numbers a from the extended complex plane.

§ 2. The deficiency quantities and other asymptotic properties of meromorphic functions of lower order $\lambda < 1$ have been studied by many authors [3-6].

We note the following important assertion [3, 4].

Theorem B. If $f(z)$ is a meromorphic function of lower order $0 \leq \lambda < 0.5$ and, for some a ,

$$\delta(a, f) > 1 - \cos \pi \lambda,$$

then

$$\lim_{r \rightarrow \infty} \mu(r, a, f) = \infty,$$

$$\mu(r, a, f) = \min_{|z|=r} \frac{1}{|f(z) - a|}.$$

It follows from this theorem that if the lower order of $f(z)$ is $\lambda < 0.5$ and $f(z)$ has no fewer than two deficient values, then all the magnitudes of the deficiencies are sufficiently small. Stronger results in this direction are known (see ^(3, 5)).

In the present work we obtain the following assertions on the quantities $\beta(a, f)$ for meromorphic functions of finite lower order $\lambda < 1$, which are certain analogues of the preceding results.

Theorem 1. If, for a meromorphic function $f(z)$ of finite lower order $0 \leq \lambda < 0.5$, for some a

$$\beta(a, f) > \pi \lambda \sin \pi \lambda$$

or

$$\beta(a, f) > \pi \lambda \tan \frac{\pi \lambda}{2} \cdot (2 - \delta(a, f)),$$

then

$$\overline{\lim}_{r \rightarrow \infty} \mu(r, a, f) = \infty,$$

and, consequently,

$$\beta(b, f) = 0 \text{ and } \delta(b, f) = 0$$

for all $b \neq a$.

Remark. The deviations of $f(z)$ from all numbers $b \neq a$ are also equal to zero if the weaker relations ($0 \leq \lambda \leq 0.5$)

$$\beta(a, f) \geq \pi \lambda \sin \pi \lambda$$

or

$$\beta(a, f) \geq \pi \lambda \tan \frac{\pi \lambda}{2} \cdot [2 - \delta(a, f)]$$

are satisfied.

Theorem 2. If a meromorphic function $f(z)$ of finite lower order $\lambda < 1$ has positive deviations from at least two points a and b , then

$$\beta(a, f) + \beta(b, f) \leq \pi \lambda \tan \frac{\pi \lambda}{2} \cdot [2 - \delta(a, f) - \delta(b, f)],$$

$$\beta(a, f) + \cos \pi \lambda \beta(b, f) \leq \pi \lambda \sin \pi \lambda \cdot [1 - \delta(b, f)] \quad (0 \leq \lambda < 0.5).$$

The example analyzed in the work of A. Edrei and W. Fuchs ⁽⁵⁾ shows that the assertions of Theorems 1 and 2 cannot be strengthened.

§ 3. The proofs of Theorems 1 and 2 are carried out by means of the method used by us in ⁽¹⁾. Indeed, let, for fixed r , $\theta(r)$ and $\theta^*(r)$ be defined by the condition ($0 \leq \theta(r) < 2\pi$, $0 \leq \theta^*(r) < 2\pi$)

$$|f(re^{i\theta(r)})| = M(r, \infty, f), \quad \frac{1}{|f(re^{i\theta^*(r)})|} = M(r, 0, f).$$

For fixed r , together with the meromorphic function $f(z)$ ($f(0) = 1$), consider the meromorphic functions

$$F_r(z) = f(re^{i\theta(r)}z), \quad F_r^*(z) = \frac{1}{f(e^{i\theta^*(r)}z)} \quad (z = se^{i\theta}).$$

For each fixed r ($r_0 \leq r \leq 0.5R$) we have (see ⁽¹⁾)

$$\begin{aligned} \ln M(r, \infty, f) + \ln M(r, 0, f) &\leq (2x)^2 r^{2x} \int_0^R \left\{ \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \ln |F_r(te^{i\theta})| d\theta + \right. \\ &\quad \left. + \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \ln |F_r^*(te^{i\theta})| d\theta \right\} \frac{t^{2x-1} dt}{(t^{2x} + r^{2x})^2} + \sum_{|b_k| \leq 2R} \ln \left| \frac{r^{2x} + |b_k|^{2x}}{r^{2x} - |b_k|^{2x}} \right| + \\ &\quad + \sum_{|a_k| \leq 2R} \ln \left| \frac{r^{2x} + |a_k|^{2x}}{r^{2x} - |a_k|^{2x}} \right| + C_1 \left(\frac{r}{R} \right)^{2x} \{T(4R, f) + T_1(4R, f)\}; \end{aligned} \quad (1)$$

where $x = \pi/2\alpha$, $\pi - \varepsilon < \alpha = \alpha(r) < \pi$ ($0 < \varepsilon < \pi/2$), a_k are the zeros of the meromorphic function $f(z)$, and b_k are its poles (C_1 is a positive constant).

Moreover ($0 \leq t \leq R$),

$$\int_{-\pi}^{\pi} \ln |F_r(te^{i\theta})| d\theta + \int_{-\pi}^{\pi} \ln |F_r^*(te^{i\theta})| d\theta = 0. \quad (2)$$

Theorem 2 follows from estimate (1), if one uses relation (2) and carries out arguments analogous to those given in papers ^(1, 7, 8). **Theorem 1** is proved analogously.

Theorems 1 and 2 can also be obtained by means of the method used in papers ^(3, 9).

Kharkov State University
named after A. M. Gorky

Received
12 XI 1968

REFERENCES

- ¹ V. P. Petrenko, DAN, **184**, No. 5 (1969).
- ² R. Nevanlinna, *Single-valued analytic functions*, Moscow-Leningrad, 1941.
- ³ I. V. Ostrovskii, DAN, **150**, No. 1, 32 (1963).
- ⁴ A. A. Gol'dberg, DAN, **98**, No. 3, 893 (1954).
- ⁵ A. Edrei, W. H. J. Fuchs, Duke Math. J., **27**, No. 3, 233 (1960).
- ⁶ G. Valiron, C. R., **230**, 40 (1950).
- ⁷ V. P. Petrenko, DAN, **155**, No. 2 (1964).
- ⁸ V. P. Petrenko, DAN, **158**, No. 5, 1030 (1965).
- ⁹ A. A. Gol'dberg, I. V. Ostrovskii, Notes of the Mathematical Department of Kharkov State University and the Kharkov Mathematical Society, **27**, ser. 4, 3 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.