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T. A. Sarymsakov, M. A. Mirzakhmedov

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Abstract

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MATHEMATICS

Academician of the Academy of Sciences of the Uzbek SSR T. A. SARYM-SAKOV, M. A. MIRZAKHMEDOV

STATIONARY PROCESSES ON TOPOLOGICAL SEMIFIELDS

We shall adhere to the definitions and notation of the article ⁽¹⁾.

1. Let E be a topological semifield; ∇ the topological Boolean algebra of its idempotents; m a measure defined on some maximal confinate $I^* \subset \nabla$, continuous with respect to the topology of the semifield. An element $x \in \overline{K} \cap S$, where $\overline{K}$ is the cone of nonnegative elements of E , and S is the set of all summable elements of E , will be called a probability distribution if $\mu(x) = 1$.

Introduce the notation

$$P = \{x : x \in \overline{K} \cap S, \mu(x) = 1\}.$$

Each element $x \in P$ generates a continuous probability measure μ_x , defined on $\nabla^* \subset \nabla$. A random variable in E is specified by a pair $\{\xi, x\}$, where ξ is an element measurable with respect to the measure m ⁽¹⁾, and $x \in P$.

Let a family of elements of the semifield measurable with respect to the measure m be given in E , $\{\xi_t, t \in T\}$.

Definition 1. (see ^(2,3)). The quadruple $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ is called a random process if the following (consistency) conditions are satisfied:

- 1) for any permutation $i_1, i_2, \dots, i_n$ of the numbers $1, 2, \dots, n$,

$$\mu_{x_{t_{i_1}}, t_{i_2}, \dots, t_{i_n}} = \mu_{x_{t_1}, t_2, \dots, t_n};$$

- 2) for any $t_1, t_2, \dots, t_k$ and any $\lambda_1, \lambda_2, \dots, \lambda_k$ ($k < n$),

$$\mu_{x_{t_1}, t_2, \dots, t_n} \left(\bigwedge_{i=1}^k e_{\lambda_i}^{\xi_{t_i}} \right) = \mu_{x_{t_1}, t_2, \dots, t_k} \left(\bigwedge_{i=1}^k e_{\lambda_i}^{\xi_{t_i}} \right),$$

where

$$e_{\lambda}^{\xi} = \bigvee_{e \in I_{\nabla}} \{e : \xi e \leq \lambda e\}.$$

Sometimes, for simplicity, instead of $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ we shall write ξ_t . Along with the semifield E , we introduce into consideration the complex semifield $\mathcal{E}$, defined by the equality $\mathcal{E} = E + iE$.

Definition 2. Let $\mu(\xi_t x) = \varphi(t)$,

$$\mu[(\xi_t - \varphi(t))(\xi_s - \varphi(s))x_{t,s}] = v(t, s).$$

If for all t, s, h , $\varphi(t+h) = \varphi(t)$ and $v(t+h, s+h) = v(t, s)$, then ξ_t is called a stationary process in the broad sense. If for all $t_1, t_2, \dots, t_n, h$

$$\mu_{x_{t_1+h, t_2+h, \dots, t_n+h}} \left(\bigwedge_{i=1}^n e_{\lambda_i}^{\xi_{t_i+h}} \right) = \mu_{x_{t_1, t_2, \dots, t_n}} \left(\bigwedge_{i=1}^n e_{\lambda_i}^{\xi_{t_i}} \right),$$

then ξ_t is called a stationary process in the narrow sense.

The following theorem is easily proved.

Theorem 1. If the random process ξ_t is stationary in the narrow sense and $\mu(\xi_0^2 x) < \infty$, then it is stationary also in the broad sense.

If the Gaussian random process ξ_t is stationary in the broad sense, then it is stationary also in the narrow sense.

2. Let us consider the question of the spectral representation of a stationary process in the broad sense.

Definition 3. We shall call a decomposition of the identity of the semifield E a one-parameter family of idempotents e_{λ} satisfying the conditions:

- 1) $e_{\lambda} \cdot e_{\gamma} = e_{\lambda}$, $\lambda = \min\{\lambda, \gamma\}$;
- 2) $e_{\lambda \rightarrow 0} = e_{\lambda}$ (in the sense of the topology of the semifield);
- 3) $e_{-\infty} = 0$, $e_{\infty} = 1$, the identity of the semifield.

It is known (4) that any element $z \in E$ has the integral representation

$$z = \int_{-\infty}^{\infty} \lambda de_{\lambda}^z, \quad (1)$$

where e_{λ}^z is the decomposition of the identity generated by the element z . For the theory of random processes it is important that any process be representable in the form of an integral with respect to some decomposition of the identity.

Let ξ be a measurable element of the semifield and e some idempotent. Denote by $\beta_\xi e$ the inf of the set of those numbers β for which $\xi e \leq \beta e$ on the whole idempotent e . Analogously, $\alpha_\xi e$ means the sup of the set of those numbers α for which $\xi e \geq \alpha e$.

Form the sums

$$\begin{aligned} s_n &= \sum_{i=1}^n \alpha_{\xi_i}^{(i)} \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) \leq \sum_{i=1}^n \xi_t \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) \leq \\ &\leq \sum_{i=1}^n \beta_{\xi_t}^{(i)} \cdot (e_{\lambda_{i+1}} - e_{\lambda_i}) = S_n. \end{aligned}$$

Naturally, now in representation (1), instead of λ there will be a function of λ and t . Suppose that the following is fulfilled.

Condition A. As $\max |\lambda_{i+1} - \lambda_i| \rightarrow 0$, the sums s_n and S_n have one common limit.

When condition A is fulfilled, we shall write

$$\xi_t = \int_{-\infty}^{\infty} f_{\xi_t}(\lambda) de_\lambda \equiv \int_{-\infty}^{\infty} f(\lambda, t) de_\lambda. \quad (2)$$

The representation of the process in the form (2) is unique. Indeed, let

$$\xi'_t = \int_{-\infty}^{\infty} f(\lambda, t) de'_\lambda$$

be another representation of ξ_t . If $f(\lambda, t) = 0$, then the uniqueness of (2) is clear.

Suppose that $f(\lambda, t) \neq 0$ and $de_\lambda \neq de'_\lambda$. Then

$$\xi_t - \xi'_t = \theta = \int_{-\infty}^{\infty} f(\lambda, t) d(e_\lambda - e'_\lambda) \neq 0,$$

which leads to a contradiction. Thus, the following has been proved.

Theorem 2. *If condition A is fulfilled, then for any random process $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ and any decomposition of the identity e_λ , there exists a unique function $f(\lambda, t)$ for which*

$$\xi_t = \int_{-\infty}^{\infty} f(\lambda, t) de_\lambda.$$

Theorem 3. A random process $\{E, \nabla^*, \mu, \xi_t, t \in T\}$ is representable in the form (2) if and only if its correlation function is representable in the form

$$\mu(\xi_t \bar{\xi}_s x) = \int_{-\infty}^{\infty} f(\lambda, t) \bar{f}(\lambda, s) dF(\lambda).$$

The proof of the theorem follows from the relations:

1)

$$\mu(((de_\lambda)x)) = \mu(\overline{(de_\lambda)x}) = \mu(((e_{\lambda+d\lambda} - e_\lambda)x)) = F(\lambda+d\lambda) - F(\lambda) = dF(\lambda);$$

2) for any functions $f(\lambda)$ and $g(\lambda)$ that are square-integrable with respect to the measure $dF(\lambda)$,

$$\mu\left(\left(\int_{-\infty}^{\infty} f(\lambda) de_\lambda \int_{-\infty}^{\infty} \overline{g(\lambda)} de_\lambda\right)x\right) = \int_{-\infty}^{\infty} f(\lambda) \overline{g(\lambda)} dF(\lambda).$$

If in (2) $f(\lambda, t) = e^{i\lambda t}$, then the random process is stationary in the wide sense.

Corollary 1. The random process ξ_t is representable in the form

$$\xi_t = \int_{-\infty}^{\infty} e^{i\lambda t} de_\lambda \quad (3)$$

if and only if its correlation function is representable in the form

$$\mu(\xi_t \bar{\xi}_s x) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} dF(\lambda).$$

In view of the uniqueness of the representation (2), every stationary process in the wide sense admits the spectral decomposition (3).

Corollary 2. The random process ξ_t is stationary in the wide sense if and only if

$$\xi_t = \int_{-\infty}^{\infty} e^{i\lambda t} de_\lambda.$$

Let us note that, with an appropriate choice of the semifield, from (3) we obtain the spectral representations of an ordinary stationary process and of a homogeneous random field.

Tashkent State University
named after V. I. Lenin

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Note: Figure translations are in progress. See original paper for figures.

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