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A. T. Fomenko

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Abstract

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MATHEMATICS

A. T. FOMENKO

EXISTENCE AND ALMOST EVERYWHERE REGULARITY OF MINIMAL COMPACTS WITH PRESCRIBED HOMOLOGICAL PROPERTIES

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1. An essential advance in the solution of the multidimensional Plateau problem was the theorem on the existence and almost everywhere regularity of a compact set X_0 realizing the absolute minimum of the Hausdorff measure $\Lambda^k(X \setminus A)$ over all compact sets X "spanning," in a certain precise sense, the compact set $A \subset E^n$, where E^n is Euclidean space. This theorem, proved by Reifenberg ⁽¹⁻³⁾, was generalized by Morrey to the case of Riemannian manifolds ^(4,5). The possibility of these results lies in the new definition of boundary proposed by J. F. Adams and Reifenberg and carefully studied by Adams in the "Appendix" to ⁽¹⁾.

In the present note a theorem is formulated on the existence and almost everywhere regularity of a minimal compact set in the class of all compact sets realizing a subgroup $\mathcal{L}'$ in $H_k(\mathfrak{M}^n)$, where $\mathfrak{M}^n$ is a compact Riemannian manifold.

2. By $H_*(X, \mathfrak{G})$ we shall denote the Čech-Aleksandrov homology of the space X with coefficients in the Abelian group $\mathfrak{G}$.

Definition 1. Let $k > 1$, and let A be an arbitrary but fixed compact set in $\mathfrak{M}^n$; $X \subset \mathfrak{M}^n$. The **algebraic boundary** $b(X, A, \mathfrak{G})$ of the compact set $X \supset A$ with respect to A in dimension k is the group $\text{Ker } i_*$, where

$$i_* : H_{k-1}(A, \mathfrak{G}) \rightarrow H_{k-1}(X, \mathfrak{G})$$

is the homomorphism induced by the inclusion $i : A \rightarrow X$.

Definition 2. Let $\mathcal{L}$ be a subgroup in $H_{k-1}(A)$; $\mathcal{L} \neq 0$; $A \subset \mathfrak{M}^n$. The **class** $\mathfrak{G}^k(A, \mathcal{L})$ is the totality of all compact sets $X \subset \mathfrak{M}^n$ such that $b(X, A, \mathfrak{G}) \supset \mathcal{L}$. (The case $\mathcal{L} = 0$ is uninteresting.)

Remark. Morrey's existence theorem is the existence theorem in the class $\mathfrak{G}^k(A, \mathcal{L})$.

3. It turns out that in $\mathfrak{G}^k(A, \mathcal{L})$ it is reasonable to distinguish subclasses "parametrized" by subgroups $\mathcal{L}' \subset H_k(\mathfrak{M}^n, \mathfrak{G})$.

Definition 3. Let $\mathcal{L}' \subset H_k(\mathfrak{M}^n, \mathfrak{G})$; $A \subset \mathfrak{M}^n$. To the class

$$O^k(A, \mathcal{L}, \mathcal{L}')$$

we assign all compact sets $X \in \mathfrak{G}^k(A, \mathcal{L})$ such that $\alpha_* H_k(X) \supset \mathcal{L}'$, where α_* is the homomorphism induced by the inclusion $\alpha: X \rightarrow \mathfrak{M}^n$.

Ignoring the presence of $\mathcal{L}'$, i.e., formally putting $\mathcal{L}' = 0$, we obtain the equality $O^k(A, \mathcal{L}, 0) = \mathfrak{G}^k(A, \mathcal{L})$; ignoring the existence of $\mathcal{L}$, i.e., formally putting $\mathcal{L} = 0$ (which was meaningless in $\mathfrak{G}^k(A, \mathcal{L})$), we obtain classes $O^k(A, 0, \mathcal{L}')$, whose elements are connected with A only by the inclusion relation $X \supset A$; but if $A = \emptyset$, the classes $O_k(\emptyset, 0, \mathcal{L}')$ consist of compact sets "freely" realizing $\mathcal{L}'$.

4. **Main theorem.** Let $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p , where $p \geq 4$; let A be an arbitrary compact set in $\mathfrak{M}^n$; let $\mathfrak{G}$ be either a compact Abelian group or a finite-dimensional vector space over some field F ; let k be an integer and $k \geq 3$; and let $\mathcal{L}$ and $\mathcal{L}'$ be subgroups in $H_{k-1}(A, \mathfrak{G})$ and $H_k(\mathfrak{M}^n, \mathfrak{G})$, respectively, at least one of them being nontrivial. Suppose that $O^k(A, \mathcal{L}, \mathcal{L}') \neq \emptyset$ and that $d(A, \mathcal{L}, \mathcal{L}') < \infty$, where

$$d(A, \mathcal{L}, \mathcal{L}') = \inf \Lambda^k(X \setminus A); \quad X \in O^k(A, \mathcal{L}, \mathcal{L}').$$

Then there exists a compact set $X_0 \in O^k(A, \mathcal{L}, \mathcal{L}')$ such that $\Lambda^k(X_0 \setminus A) = d(A, \mathcal{L}, \mathcal{L}')$, and every point $x \in (X_0 \setminus A) \setminus Z$, where $\Lambda^k(Z) = 0$, has in $X_0 \setminus A$ a neighborhood homeomorphic to a k -dimensional disk D^k . Moreover, if $\mathfrak{M}^n \in C^4$, then these k -disks may be assumed to belong to the class C_μ^3 for any $0 < \mu < 1$; if $\mathfrak{M}^n \in C_\mu^p$ for some $p \geq 4$ and some $0 < \mu < 1$, then the k -disks may be assumed to belong to the class C_μ^p ; finally, if $\mathfrak{M}^n \in C^\infty$ or is analytic, then the k -disks may be assumed to belong to the class C^∞ or to be analytic, respectively.

Putting $\mathcal{L}' = 0$, we obtain the theorem of Morrey; if, however, $A = \emptyset$, we obtain an existence theorem for "free cycles."

5. In the proof the following three theorems are used:

Theorem 1. Each class $O^k(A, \mathcal{L}, \mathcal{L}')$ is closed with respect to pointwise convergence $\lim_{n \rightarrow \infty} X_n$. By $\lim_{n \rightarrow \infty} X_n$ we mean the set of limit points of all possible sequences $\{x_n\}$, where $x_n \in X_n$, $n = 1, 2, \dots$

Theorem 2 (Adams). Let $X \in \mathfrak{G}^k(A, \mathcal{L})$, G be open in $\mathfrak{M}^n$; $A \cap \overline{G} = \emptyset$; $X \cap \partial G = B$; U be compact, $U \subset \overline{G}$; $U \cap \partial G = B$; $b(U, B) \supset b(X_1, B)$, where

$X_1 = X \cap \overline{G}$. Then the compact set X' , obtained from X by replacing $X \cap G$ by $U \cap G$, again belongs to $\mathfrak{G}^k(A, \mathcal{L})$.

Theorem 3 (Adams). Let $A = A_1 \cup A_2$, where A_1 and A_2 are compact sets, $A_1 \cap A_2 = D$, $C_1 = A_1 \cup B$, $C_2 = A_2 \cup B$, $C = C_1 \cup C_2 = A \cup B$, where B is such a compact set that $b(B, D) \supset \overline{H}_{k-1}(D)$, $k \geq 3$. Then

$$i(C, C_1)_* H_k(C_1) + i(C, C_2)_* H_k(C_2) \supset i(C, A)_* H_k(A),$$

where $i(X, Y)$ is the embedding $Y \rightarrow X$.

The main observation consists in the fact that Theorem 3 can be applied to prove the closedness of the classes $O^k(A, \mathcal{L}, \mathcal{L}')$ with respect to deformations of a special kind, namely: cutting off long “whiskers” of the compact set X , which have little effect on the measure Λ^k , and replacing them by flat caps.

6. The construction of the minimal compact set X_0 is carried out as follows. Let $R(P) = \min(\rho(P, A), R_0)$, where P is a point in $\mathfrak{M}^n$; $\rho(x, y)$ is the distance in $\mathfrak{M}^n$; R_0 is some positive number independent of the point. Define

$$\varphi(r, P, X) = \int_0^r \Lambda^{k-1}[X \cap \partial B(P, t)] dt; \quad 0 < R(P), \quad 0 < r < R(P);$$

$B(P, t)$ is the open ball of radius t with center at P ; $\psi(r, P, X) = \Lambda^k[X \cap \overline{B}(P, r)]$.

Let $\{X_n\}$ be a minimizing sequence; $X_n \in O^k(A, \mathcal{L}, \mathcal{L}')$, $\Lambda^k(X_n \setminus A) = d(A, \mathcal{L}, \mathcal{L}') + \varepsilon_n$; $\{Q_i\}$ is a countable set dense in $\mathfrak{M}^n$. Then there exists a subsequence $\{X_{n'}\}$ such that the functions $\psi(r, Q_i, X_{n'})$ converge for every i and every r , $0 < r < R(P)$, to functions $\tilde{\psi}(r, Q_i)$. Put

$$\psi^+(r, Q) = \lim_{\varepsilon \rightarrow 0} \sup_{\substack{r' < r + \varepsilon \\ \rho(Q, Q_i) < \varepsilon}} \tilde{\psi}(r', Q_i); \quad \psi^-(r, Q) = \lim_{\varepsilon \rightarrow 0} \inf_{\substack{r' > r - \varepsilon \\ \rho(Q, Q_i) < \varepsilon}} \tilde{\psi}(r', Q_i);$$

$$\varphi^+(r, Q) = \limsup_{n \rightarrow \infty} \varphi(r, Q, X_n); \quad \varphi^-(r, Q) = \liminf_{n \rightarrow \infty} \varphi(r, Q, X_n).$$

Let

$$\psi(P) = \lim_{r \rightarrow +0} [h(r)]^{-1} \psi^+(r, P), \quad h(r) = \alpha(k) r^k (1 + hr)^{-k};$$

$\alpha(k)$ is the volume of the unit k -dimensional ball; h is some constant. Then, it turns out, $X_0 = S \cup A$, where $S = \{P \mid P \in \mathfrak{M}^n \setminus A, \psi(P) > 0\}$.

7. The proof of the main theorem is led by a chain of assertions, the most important of which are:

Theorem 4. Let $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p , $p \geq 4$, and $H_k(\mathfrak{M}^n) \neq 0$. Let $d' = \inf \Lambda^k(X)$, where $j_* H_k(X) \neq 0$; $j : X \rightarrow \mathfrak{M}^n$. Then $d' > 0$.

Theorem 5. The sequence $\{X_n\}$ can be modified in such a way that the new sequence $\{\tilde{X}_n\}$ has the very same functions

$\psi^\pm, \tilde{X}_n \in O^k(A, \mathcal{L}, \mathcal{L}')$, while the new functions $\varphi^\pm$ do not decrease in r , and $\bar{\rho}(\tilde{X}_n, X_0) \rightarrow 0$ as $n \rightarrow \infty$, where

$$\bar{\rho}(\tilde{X}_n, X_0) = \max_{x \in X_0} \rho(\tilde{X}_n, x) + \max_{y \in \tilde{X}_n} \rho(y, X_0).$$

The proof of local differentiability uses Theorem 3.

8. Corollary 1. Consider the class $R^k(A)$, where $X \in R^k(A)$ if $\alpha_* H_k(X) \neq 0$; $\alpha : X \rightarrow \mathfrak{M}^n$. Let $k \geq 3$, $\mathfrak{M}^n$ be a compact Riemannian manifold of class C^p , $p \geq 4$, $A \subset \mathfrak{M}^n$, $\mathfrak{G}$ be either a finitely generated compact Abelian group or a finite-dimensional vector space over some field F . Then there exists a compact set $X'_0 \in R^k(A)$ such that

$$\Lambda^k(X'_0 \setminus A) = d'(A) = \inf_{X \in R^k(A)} \Lambda^k(X \setminus A)$$

and the local structure of this compact set is the same as for the compact sets of the main theorem. If $A \in R^k(A)$, then $d'(A) > 0$.

Consider the class $N^k(A)$, where $X \in N^k(A)$ if $b(X, A) \neq 0$. Then, if $H_{k-1}(A, \mathfrak{G})$ is either a finitely generated compact Abelian group or a finite-dimensional vector space over a field, there exists a minimal compact set $\tilde{X}_0$ with analogous properties;

$$\tilde{d}(A) = \inf_{x \in N^k(A)} \Lambda^k(X \setminus A) = \Lambda^k(\tilde{X}_0 \setminus A).$$

9. Along with the obvious relations

$$\begin{aligned} 0 < d(A, \mathcal{L}, 0) &\leq d(A, \mathcal{L}, \mathcal{L}') \geq d(A, 0, \mathcal{L}'); \\ d(A, 0, \mathcal{L}') &= 0, \quad \text{if } A \in O^k(A, 0, \mathcal{L}'); \\ d(A, 0, \mathcal{L}') &> 0, \quad \text{if } A \notin O^k(A, 0, \mathcal{L}'), \end{aligned}$$

the following holds.

Corollary 2. Let $A = f(S^{k-1})$, where f is a homeomorphism; S^{k-1} is a sphere; $\mathfrak{G} = U$, the group of real numbers modulo 1; $\mathcal{L}$ and $\mathcal{L}'$ are nontrivial subgroups and $\mathcal{L}' \supset U$. Then either

$$\tilde{d}(A) < d(A, \mathcal{L}, \mathcal{L}'),$$

or

$$d'(A) < d(A, \mathcal{L}, \mathcal{L}').$$

With the compact sets $\widetilde{X}_0$ and X'_0 one can associate a chain of groups $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$, where

$$\mathfrak{G}_1 = \text{Im } H_k(\widetilde{X}_0), \quad \mathfrak{G}_2 = \text{Im } H_k(X'_0), \quad \mathfrak{G}_3 = b(\widetilde{X}_0, A), \quad \mathfrak{G}_4 = b(X'_0, A).$$

Corollary 3. For any compact Riemannian $\mathfrak{M}^n \in C^p$, $p \geq 4$, any homeomorphism f , and $\mathfrak{G} = U$, the chain $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3, \mathfrak{G}_4)$ is isomorphic to one of the following chains:

$$(\text{finite cyclic, infinite cyclic, } \neq 0, \neq \mathfrak{G}_4),$$

$$(\mathfrak{G}_1, U, \neq 0, 0),$$

$$(0, U, \neq 0, \neq 0),$$

$$(\supseteq U, \subseteq U, \neq 0, 0).$$

10. The notion of realization can also be defined by means of the homomorphism

$$\omega_* : \overline{H}_k(\mathfrak{M}^n, A) \rightarrow \overline{H}_k(\mathfrak{M}^n, S \cup A),$$

i.e., a compact set S realizes a subgroup $P \subset \overline{H}_k(\mathfrak{M}^n, A)$ if $\omega_* P = 0$. In some particular cases (for example, if $A = \emptyset$) the ω -variant of the minimum problem and the α -variant coincide. In (6) the ω -variant is investigated in the case when $\mathfrak{G}$ is a finitely generated Abelian group; A is a $(k-1)$ -smoothable compact set; P is a subgroup generated by one element σ , and the minimum Λ^k is taken over k -smoothable compact sets.

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Mechanics and Mathematics Faculty of Moscow State University named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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