

TOPOLOGICAL ENTROPY OF A COUNTABLE MARKOV CHAIN

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Abstract

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MATHEMATICS

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TOPOLOGICAL ENTROPY OF A COUNTABLE MARKOV CHAIN

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1. Introduction. Recently several authors have considered topological analogues of Markov chains (see ⁽¹⁾). In this note, a **topological Markov chain** will mean the shift T_Γ in the space $X(\Gamma)$ of infinite (from $-\infty$ to $+\infty$) paths of a finite or countable directed graph Γ . Parry ⁽²⁾ computed the topological entropy $h_{\text{top}}(T_\Gamma)$ (see ⁽³⁾) in the case of finite Γ (in ⁽²⁾ the term “absolute entropy” is used), which turned out to be equal to $\log \lambda(\Gamma)$, where $\lambda(\Gamma)$ is the maximal eigenvalue of the adjacency matrix of the graph Γ . Another result of ⁽²⁾ is that there exists a unique T_Γ -invariant measure μ_0 for which the metric entropy $h_{\mu_0}(T_\Gamma)$ coincides with $h_{\text{top}}(T_\Gamma)$ (for the remaining measures $h_\mu(T_\Gamma) < h_{\text{top}}(T_\Gamma)$), and after introducing this measure the topological chain turns into an ordinary Markov chain with a finite number of states*. It later became clear that topological Markov chains, both finite and countable, play an important role in the construction of symbolic dynamics for certain classes of dynamical systems (see ^(1, 4-6)).

Wishing to define the topological entropy of a countable Markov chain, we encounter the noncompactness of the space $X(\Gamma)$, because of which the usual definition loses its meaning. This difficulty can be overcome by calling the following number the topological entropy:

$$h(T_\Gamma) = \sup_{\Gamma'} \log \lambda(\Gamma') = \sup_{\Gamma'} h_{\text{top}}(T_{\Gamma'}),$$

where the supremum is taken over all finite subgraphs of the graph Γ . The quantity $h(T_\Gamma)$ is convenient for computation, and it is easy to construct a sequence $\{\mu_n\}$ of T_Γ -invariant measures for which

$$h_{\mu_n}(T_\Gamma) \xrightarrow{n \rightarrow \infty} h(T_\Gamma).$$

Another natural path is to compactify the space $X(\Gamma)$, allowing one then to apply the usual definition of topological entropy. It will be shown below that both paths lead to one and the same result. Thus, the assertion of ⁽⁶⁾, concerning

chains of a special form, turns out to be valid in the general case. Along the way one discovers a fact which, apparently, has not previously been observed: there exists a homeomorphism T of a compact metric space such that for every T -invariant measure μ ,

$$h_\mu(T) < h_{\text{top}}(T).$$

A corresponding example can be provided by the topological Markov chain described in Section 3.

2. Main result. Identify the set of vertices of the graph Γ under consideration with the set Z of natural numbers. We shall assume that the graph Γ is **connected**, i.e., for any two of its vertices i, j there exists a path leading from i to j . The number of edges entering a path γ is called its **length**. The **adjacency matrix** $M(\Gamma) = (m_{i,j}(\Gamma))$ is defined

* All these results are repeated in ⁽⁴⁾.

by the equality

$$m_{ij}(\Gamma) = \begin{cases} 1, & \text{if there exists a path of length 1 from } i \text{ to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The set $X(\Gamma)$ consists of all infinite (in both directions) sequences $\{i_n\}$ for which $m_{i_n i_{n+1}} = 1$ for every $n = 0, \pm 1, \pm 2, \dots$. From the definition it is clear that $X(\Gamma)$ is invariant with respect to the shift taking the sequence $\{i_n\}$ into $\{i'_n\}$, where $i'_n = i_{n-1}$. Complete the set Z with the discrete topology by the point ∞ so as to obtain the compact space $\overline{Z}$. Then the space $\overline{X} = \prod_{-\infty < n < \infty} \overline{Z}_n$, where $\overline{Z}_n = \overline{Z}$, is also compact and $X(\Gamma) \subset \overline{X}$. On $\overline{X}$ one can introduce a metric by setting (see (6))

$$\rho(\{i_n\}\{j_n\}) = \sum_{-\infty < n < \infty} \frac{1}{2^{|n|}} \frac{1 - \delta_{i_n, j_n}}{\min(i_n, j_n)}; \quad i_n, j_n = 1, 2, \dots, \infty$$

(it is assumed that $\min(i, \infty) = i$, $\min(\infty, \infty) = \infty$, $1/\infty = 0/\infty = 0$).

The closure $\overline{X(\Gamma)}$ of the set $X(\Gamma)$ in $\overline{X}$ is a compact metric space, and the shift T_Γ is its homeomorphism.

Theorem. $h(T_\Gamma) = h_{\text{top}}(T_\Gamma)$.

The proof of the theorem is based on several auxiliary assertions.

Lemma 1. *For any connected graph Γ there exists a sequence $\{\Gamma_k\}$ of its finite connected subgraphs such that: 1) Γ_k is a subgraph of the graph Γ_{k+1} , $k = 1, 2, \dots$; 2) if $i \leq k^2$, then i belongs to the set $G(\Gamma_k)$ of vertices of the graph Γ_k . **

Lemma 2. *Let G be an arbitrary finite set of vertices of the connected graph Γ . Then there exist a connected subgraph $\Gamma'(G)$ of the graph Γ and a number $l'(G)$ such that, for any path in Γ of length $\geq l'(G)$ with beginning and end in G , there is a path in $\Gamma'(G)$ of the same length and with the same beginning and end.*

Remark. From this formulation it is clear that G is contained in the set of vertices of the graph $\Gamma'(G)$.

Lemma 3. *If G is any finite set of vertices of a graph Γ satisfying the condition $h(T_\Gamma) < \infty$, and l is any positive number, then in Γ there can be found only finitely many paths of length $\leq l$ beginning and ending in G .*

Lemma 4. *Let $P_n(r)$ be the number of sequences of length n of zeros and ones in which every run of zeros has length $\geq r$. Then*

$$\lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{n} \log P_n(r) = 0.$$

Sketch of the proof of the theorem. Let ξ_k be the partition of the space $\overline{X(\Gamma)}$, defined as follows: the points $\{i_n\}$ and $\{j_n\}$ belong to one element C_ξ of the partition ξ_k if and only if $i_0 \notin G(\Gamma_k)$, $j_0 \notin G(\Gamma_k)$, or $i_0 \in G(\Gamma_k)$, $j_0 \in G(\Gamma_k)$, $i_0 = j_0$. The partition ξ_k is at the same time an open cover. Put

$$\eta_k = \bigvee_{i=-k}^k T_\Gamma^i \xi_k.$$

By Lemma 1,

$$\eta_k \leq \eta_{k+1}, \quad \max_{C_{\eta_k}}(\text{diam } C_{\eta_k}) \xrightarrow{k \rightarrow \infty} 0.$$

It follows from this (see (3)) that

$$\lim_{k \rightarrow \infty} h_{\text{top}}(T_\Gamma, \xi_k) = \lim_{k \rightarrow \infty} h_{\text{top}}(T_\Gamma, \eta_k) = h_{\text{top}}(T_\Gamma). \quad (1)$$

* Any subgraph of a given graph is uniquely determined by the set of its vertices.

Let us further note that the sequence $\{\Gamma_k\}$ described in Lemma 1 necessarily has the property that any finite subgraph Γ' of the graph Γ is a subgraph of some Γ_k . In this case $\lambda(\Gamma') \leq \lambda(\Gamma_k)$ and, consequently,

$$\lim_{k \rightarrow \infty} \log \lambda(\Gamma_k) = h(T_\Gamma). \quad (2)$$

Let now $N_n(\Gamma_k)$ be the number of paths of length n in the graph Γ_k , and let $N_n(\xi_k)$ be the number of elements of the partition (covering)

$$\xi_k^n = \bigvee_{i=0}^{-n} T_{\Gamma}^i \xi_k.$$

From (1), (2), using the obvious inequality $N_n(\xi_k) \geq N_n(\Gamma_k)$ and the fact that

$$h_{\text{top}}(T_{\Gamma}, \xi_k) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \log N_n(\xi_k), \quad \log \lambda(\Gamma_k) = \lim_{n \rightarrow \infty} \log \frac{1}{n+1} N_n(\Gamma_k),$$

we obtain

$$h_{\text{top}}(T_{\Gamma}) \geq h(T_{\Gamma}).$$

In proving the reverse inequality, it suffices to restrict ourselves to the case $h(T_{\Gamma}) < \infty$. Fix an arbitrary k and, applying Lemma 2 to the set $G_k = G(\Gamma_k)$, obtain a subgraph $\Gamma'(G_k)$ and a number $l'(G_k)$. Next fix any number $r \geq l'(G_k)$ and, using Lemma 3 with $G = G_k$, $l = r$, extend the subgraph $\Gamma'(G_k)$ to a finite connected subgraph $\Gamma''(G_k) = \Gamma''$, possessing the following properties: 1) every path in Γ of length $\leq r$ with beginning and end in G_k is a path in Γ'' ; 2) for every path in Γ of length $> r$ with beginning and end in G_k , there is a path of the same length in Γ'' with the same beginning and end.

These two properties of the graph Γ'' make it possible to prove the fundamental inequality

$$N_n(\xi_k) \leq 2r(r+1)N_n(\Gamma'')P_{n+1}(r), \quad (3)$$

where $N_n(\Gamma'')$ is the number of paths of length n in the graph Γ'' , and $P_{n+1}(r)$ is defined in the condition of Lemma 4. From (3) it follows that

$$\begin{aligned} h_{\text{top}}(T_{\Gamma}, \xi_k) &\leq \lim_{n \rightarrow \infty} \frac{1}{n+1} \log(2r(r+1)) + \lim_{n \rightarrow \infty} \frac{1}{n+1} \log N_n(\Gamma'') \\ &\quad + \lim_{n \rightarrow \infty} \frac{1}{n+1} \log P_{n+1}(r) \leq \log \lambda(\Gamma'') + \lim_{n \rightarrow \infty} \frac{1}{n+1} \log P_{n+1}(r) \\ &\leq h(T_{\Gamma}) + \lim_{n \rightarrow \infty} \frac{1}{n+1} \log P_{n+1}(r). \end{aligned} \quad (4)$$

Using Lemma 4, one can choose r from the very beginning so that the second term on the right-hand side of inequality (4) does not exceed a preassigned ε . This means that

$$h_{\text{top}}(T_{\Gamma}) \leq h(T_{\Gamma})$$

and, consequently,

$$h_{\text{top}}(T_\Gamma) = h(T_\Gamma).$$

3. **Example.** Consider the topological Markov chain corresponding to a graph Γ in whose adjacency matrix the only nonzero elements are

$$m_{11}(\Gamma), m_{i,i+1}(\Gamma), m_{i+1,i}(\Gamma), \quad i = 1, 2, \dots.$$

In this case the closure $\bar{X}(\Gamma)$ is obtained from $\dot{X}(\Gamma)$ by adjoining a single point $x_\infty = (\dots, \infty, \infty, \infty, \dots)$. If Γ_k is the subgraph with vertices $1, 2, \dots, k$, then, as is not hard to verify,

$$2 - \frac{1}{k} \leq \lambda(\Gamma_k) \leq 2.$$

On the basis of the theorem of Section 2, it follows from this that

$$h_{\text{top}}(T_\Gamma) = h(T_\Gamma) = \log 2.$$

We shall show that for any measure μ invariant with respect to T_Γ ,

$$h_\mu(T_\Gamma) < \log 2.$$

If this is not so, then there exists a measure μ_0 such that $h_{\mu_0}(T_\Gamma) = \log 2$. Without loss of generality, one may assume that $\mu_0(x_\infty) = 0$. The well-known theorems of the entropy theory of dynamical systems (see (7)) lead to the relations

$$\lim_{k \rightarrow \infty} h_{\mu_0}(T_\Gamma, \xi_k) = \lim_{k \rightarrow \infty} H_{\mu_0}(\xi_k | \xi_{\bar{k}}) = \lim_{k \rightarrow \infty} \int_{X(\Gamma)} H_{\mu_0}(\xi_k | C_{\xi_{\bar{k}}}(x)) d\mu_0 = \log 2. \quad (5)$$

(Here $C_{\xi_{\bar{k}}}(x)$ is the element of the partition

$$\xi_{\bar{k}} = \bigvee_{i < 0} T^i \xi_k$$

that contains the point x .)

Since any element $C_{\xi_{\bar{k}}}$ can intersect only two elements of the partition ξ_k , i.e. $H_{\mu_0}(\xi_k | C_{\xi_{\bar{k}}}(x)) \leq \log 2$, it follows from (5) that

$$H_{\mu_0}(\xi_k | C_{\xi_{\bar{k}}}(x)) \xrightarrow{\text{in measure}} \log 2. \quad (6)$$

Let E_s be the set of points $x = \{i_k\}$ for which $i_0 = s$. The collection of sets E_s , $s = 1, 2, \dots$, forms a certain partition (mod 0) ξ of the space $X(\Gamma)$. Put

$$p_1(x) = \begin{cases} 1/2, & \text{if } x \in T_\Gamma^{-1}(E_1 \cup E_2), \\ 0, & \text{otherwise;} \end{cases} \quad p_s(x) = \begin{cases} 1/2, & \text{if } x \in T_\Gamma^{-1}(E_{s-1} \cup E_{s+1}), \\ 0, & \text{otherwise;} \end{cases}$$

$$s = 2, 3, \dots$$

Using standard techniques, from (6) it is easy to derive that for almost all $x \in X(\Gamma)$

$$\mu_0(E_s \mid C_{\xi^-}(x)) = \lim_{k \rightarrow \infty} \mu_0(E_s \mid C_{\xi_k}(x)) = p_s(x), \quad s = 1, 2, \dots$$

But this means that μ_0 is a Markov measure corresponding to the transition matrix $P = (p_{ij})$, where

$$p_{ij} = \begin{cases} 1/2, & \text{if } i = j = 1 \text{ or } |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

At the same time it is known (see, for example, (8)) that a Markov chain with such a transition matrix has no finite invariant measure.

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