

ON A CLASS OF HIGH-ORDER DIFFERENTIAL- OPERATOR EQUATIONS

Yu. A. Dubinskii

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.41697>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.941.9+517.946.9

MATHEMATICS

Yu. A. DUBINSKII

ON A CLASS OF HIGH-ORDER DIFFERENTIAL-OPERATOR EQUATIONS

(Presented by Academician I. G. Petrovskii, 11 IV 1969)

This paper studies a certain class of differential-operator (d.-o.) equations (1), called the class of partially hyperbolic d.-o. equations.* The cases $s = 1, 2$ have been well studied; surveys of the main results and detailed bibliographies are available, for example, in ^(1,2). Here we consider general boundary-value problems for d.-o. equations (1) of arbitrary order $s \geq 1$.

Let H be a complex separable Hilbert space, and let $u(t) : I \rightarrow H$ be a function of the real variable $t \in I \subset \mathbb{R}^1$, taking values in H . Let, further, $A : H \rightarrow H$ be a linear operator.

Consider the d.-o. equation

$$\mathfrak{A}(u) \equiv P_s \left(\frac{d}{dt} \right) u + Au = h(t), \quad (1)$$

where

$$P_s \left(\frac{d}{dt} \right) u \equiv \sum_{q=0}^s a_q u^{(q)}(t), \quad a_q \in C^1, \quad u^{(q)} \equiv d^q u / dt^q.$$

Definition. The operator $\mathfrak{A}(u)$ is called **partially hyperbolic** if the following conditions are satisfied: 1) if s is odd, then $a_s = i$ and the operator A is self-adjoint; 2) if $s = 2l$, then $a_s = (-1)^{l-1}$ and the operator A is self-adjoint and semibounded from below.

Notation. a) $H(l, 0; \gamma)$, $l \geq 0$, denotes the space of functions $u(t) : I \rightarrow H$ having finite norm

$$\|u\|_{l,0;\gamma}^2 = \int_I (\|u^{(l)}\|^2 + \|u\|^2) e^{-\gamma t} dt,$$

where $\|u\| \equiv \|u(t)\|$ is the norm in H , and $\gamma \in \mathbb{R}^1$ is a parameter. b) $H(s, A; \gamma)$ denotes the space of functions $u(t) : I \rightarrow H$ having finite norm

$$\|u\|_{s,A;\gamma}^2 \equiv \|u\|_{s,0;\gamma}^2 + \|Au\|_{0,0;\gamma}^2.$$

- c) If A is semibounded from below, then λ_0 denotes its first eigenvalue; if A is not bounded from below, then

$$\lambda_0 = \inf_{u \in D_A} \frac{|(Au, u)|}{\|u\|^2},$$

where D_A is the domain of definition of the operator Au .

I. The case of an operator A with point spectrum

§ 1. D.-o. equations on the entire axis (the case $I \equiv \mathbb{R}^1$)

Theorem 1. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic. Then for every $\gamma \neq 0$ there exists a number $a(\gamma) \geq 0$ such that, for $\lambda_0 \geq a(\gamma)$, equation (1) has a unique solution $u(t) \in H(s, A; \gamma)$ for every right-

* Boundary-value problems for parabolic and elliptic equations of the form (1) are considered in ⁽³⁾.

part $h(t) \in H(1, 0; \gamma)$. In this case the estimate is valid

$$\|u\|_{s,A;\gamma} \leq K(\gamma) \|h\|_{1,0;\gamma}, \quad (2)$$

where $K(\gamma) > 0$ is a constant.

Corollary 1. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then equation (1) is normally solvable in the space $H(s, A; \gamma)$. This means that, for any function $h(t) \in H(1, 0; \gamma)$ subject to a finite number of conditions*, equation (1) is solvable in $H(s, A; \gamma)$. The homogeneous equation ($h(t) \equiv 0$) has a finite number of linearly independent solutions.

§ 2. Differential-operator equations on the half-axis (the case $I = R_+ \equiv [0, \infty)$).

Formulation of the problem. For a given function $h(t)$, it is required to find a solution of equation (1) satisfying, at $t = 0$, the conditions

$$R_\nu \left(\frac{d}{dt} \right) u \Big|_{t=0} = 0, \quad \nu = 0, \dots, s-1-k, \quad (3)$$

where $k = [s/2]$ if s is odd, and $k = s/2 - 1$ if s is even (**problem I**); or $k = [s/2] + 1$ for any s (**problem II**).

Here

$$R_\nu \left(\frac{d}{dt} \right) u \equiv \sum_{q=0}^{n_\nu} p_{\nu q} u^{(q)}(t), \quad p_{\nu q} \in C^1, \quad p_{\nu n_\nu} = 1, \quad n_\nu < s.$$

Suppose that the following conditions are satisfied.

Condition B. If $\lambda > 0$, then $\det \|\mu_j^{n_\nu}(\lambda)\| \neq 0$ ($\nu, j = 0, 1, \dots, s-1-k$), where $\mu_j(\lambda)$ are roots of the equation $a_s \mu^s + \lambda = 0$ such that $\operatorname{Re} \mu_j(\lambda) \leq 0$ (for problem I).

Condition C. If $\lambda > 0$, then $\det \|\mu_j^{n_\nu}(\lambda)\| \neq 0$ ($\nu, j = 0, \dots, s-1-k$), where $\mu_j(\lambda)$ are roots of the equation $a_s \mu^s + \lambda = 0$ such that $\operatorname{Re} \mu_j(\lambda) < 0$ (for problem II).

Theorem 2. Let $\gamma > 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let $\lambda_0 \geq 0$ be sufficiently large ($\lambda_0 \geq \alpha(\gamma)$). Then, if Condition B is satisfied, for any right-hand side $h(t) \in H(1, 0; \gamma)$ there exists a unique solution $u(t) \in H(s, A; \gamma)$ of problem I. In this case the estimate (2) is valid.

Theorem 3. Let $\gamma < 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let $\lambda_0 \geq \alpha(\gamma)$. Then, if Condition C is satisfied, for any right-hand side $h(t) \in H(1, 0; \gamma)$ there exists a unique solution $u(t) \in H(s, A; \gamma)$ of problem II. In this case the estimate (2) is valid.

Corollary 2. Let $\gamma > 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let Condition B be satisfied. Then problem I is normally solvable, i.e., has a finite-dimensional kernel and is solvable under a finite number of conditions on $h(t)$. In this case the inequality is valid

$$\|u\|_{s, A; \gamma} \leq K(\gamma) (\|h\|_{1, 0; \gamma} + \|u\|_{0, 0; \gamma}), \quad (4)$$

where $K(\gamma) > 0$ is a constant.

Corollary 3. Let $\gamma < 0$, let the operator $\mathfrak{A}(u)$ be partially hyperbolic, and let Condition C be satisfied. Then problem II is normally solvable and inequality (4) is valid.

If s is even, then the following holds.

Theorem 4. If, for the operator $\mathfrak{A}(u)$, where A is self-adjoint, problem I is normally solvable for every $\gamma > 0$, and problem II for every $\gamma < 0$, then the operator A is bounded below, i.e., the operator $\mathfrak{A}(u)$ is partially hyperbolic.

§ 3. Differential-operator equations on an interval (the case $I = [0, T]$).

In this case it is required to find a solution of equation (1) under the following conditions—

* These conditions have the character of orthogonality conditions of $h(t)$ to a finite-dimensional subspace for all $\gamma \in R^1$, excluding a discrete set. This also applies to [3], where in Theorems 2.1 and 3.1 it is inaccurately stated that γ is arbitrary.

forms:

$$u(0) = 0, \dots, u^{(l)}(0) = 0, \quad u'(T) = 0, \dots, u^{(l-1)}(T) = 0, \quad s = 2l \vee 1. \quad (5)$$

(here the second group of conditions in (5) is absent if $s = 1, 2$).

Theorem 5. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and let the number λ_0 be sufficiently large. Then for any function $h(t) \in H(1, 0; 0)$ the problem (1), (5) has a unique solution $u(t) \in H(s, A; 0)$. Moreover, the inequality

$$\|u\|_{s, A; 0} \leq K \|h\|_{1, 0; 0}$$

holds, where $K > 0$ is a constant.

Corollary 4. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then the problem (1), (5) is normally solvable. Moreover, the inequality

$$\|u\|_{s, A; 0} \leq K (\|h\|_{1, 0; 0} + \|u\|_{0, 0; 0})$$

holds, where $K > 0$ is a constant.

II. The case of a self-adjoint operator A with arbitrary spectrum.

Everywhere in what follows $s = 2l$ (and, consequently, $a_s = (-1)^{l-1}$) and the operator $A : H \rightarrow H$ is an arbitrary positive self-adjoint operator, i.e. $(Au, u) \geq \lambda_0(u, u)$, $\lambda_0 > 0$.

Consider the following polynomial in the real variable τ :

$$A(\gamma; \tau) \equiv \operatorname{Re}(\gamma/2 - i\tau) [P_s(\gamma/2 + i\tau) + \lambda_0].$$

Obviously, for $\gamma > 0$ the polynomial $A(\gamma; \tau)$ is semibounded below, and for $\gamma < 0$ it is semibounded above, i.e. there exists a constant $\alpha(\gamma) \geq 0$ such that, for $\gamma > 0$ ($\gamma < 0$), the polynomial

$$A_\lambda(\gamma; \tau) \equiv \operatorname{Re}(\gamma/2 - i\tau) [P_s(\gamma/2 + i\tau) + \lambda + \lambda_0] > 0 \quad (< 0)$$

for any $\lambda \geq \alpha(\gamma)$.

§ 1. D.-o. equations on the whole axis $\mathbb{R}^1$. Consider the equation

$$\mathfrak{A}_\lambda(u) \equiv \mathfrak{A}(u) + \lambda u = h(t), \quad \lambda \geq \alpha(\gamma). \quad (6)$$

Theorem 6. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and $\gamma \neq 0$. Then for any right-hand side $h(t) \in H(1, 0; \gamma)$ there exists a unique solution of equation (6), $u(t) \in H(s, A; \gamma)$. Moreover, the inequality

$$\|u\|_{s, A; \gamma} \leq K(\gamma) \|h\|_{1, 0; \gamma} \quad (7)$$

holds, where $K(\gamma) > 0$ is a constant.

§ 2. D.-o. equations on a half-axis. Given a function $h(t)$, it is required to find a solution of equation (6) under the conditions

$$u(0) = 0, \dots, u^{(l)}(0) = 0 \quad (\text{problem I}),$$

or under the conditions

$$u'(0) = 0, \dots, u^{(l-1)}(0) = 0 \quad (\text{problem II}) \quad (8)$$

(for $s = 2$ the conditions (8) are absent).

Theorem 7. Let the operator $\mathfrak{A}(u)$ be partially hyperbolic and $\gamma > 0$ ($\gamma < 0$). Then for any function $h(t) \in H(1, 0; \gamma)$ there exists a unique solution $u(t) \in H(s, A; \gamma)$ of problem I (problem II) for equation (6). Moreover, the inequality (7) holds.

§ 3. D.-o. equations on an interval. Given a function $h(t) \in H(1, 0; 0)$, it is required to find a solution of equation (6) on the interval $[0, T]$ under the conditions (5).

Theorem 8. If the operator $\mathfrak{A}(u)$ is partially hyperbolic, then for any right-hand side $h(t) \in H(1, 0; 0)$ problem (6), (5) has a unique solution $u(t) \in H(s, A; 0)$. Moreover, estimate (7) is valid for $\gamma = 0$.

§ 4. **Examples.** Let $L(x, D)u$ be an elliptic self-adjoint operator, semibounded from below, in a domain $G \subset \mathbb{R}^n$, with a coercive system of boundary conditions. In the examples we restrict ourselves to the case of the cylinder $Q = \mathbb{R}_+^1 \times G$.

1. Consider an equation of Schrödinger-equation type

$$\pm iu' + L(x, D)u = h(t, x).$$

For $t = 0$, either the initial value $u(0, x) = 0$ is prescribed (the mixed problem), or there is no initial condition (the free problem). We have

$$A(\gamma; \tau) \equiv \gamma/2 + \lambda_0,$$

whence it follows that, for $\lambda_0 \geq 0$, the mixed (free) problem is uniquely solvable for any $\gamma > 0$ ($\gamma < 0$).

2. Consider an equation of wave-equation type

$$u'' + L(x, D)u = h(t, x),$$

for which the problems studied are either the classical mixed problem or the free problem. We have

$$A(\gamma; \tau) \equiv (\gamma/2)(\gamma^2/4 + \tau^2 + \lambda_0),$$

therefore, if $\lambda_0 \geq 0$, the mixed (free) problem is uniquely solvable for any $\gamma > 0$ ($\gamma < 0$). If $\lambda_0 < 0$, unique solvability holds for

$$\gamma > 2\sqrt{|\lambda_0|} \quad (\gamma < -2\sqrt{|\lambda_0|})$$

(cf. ^(4,5)).

3. Consider an equation of the form

$$-u^{IV} + L(x, D)u = h(t, x) \quad (9)$$

under the conditions $u(0, x) = 0$, $u'(0, x) = 0$, $u''(0, x) = 0$ (problem I), or under the condition $u(0, x) = 0$ (problem II).

We have

$$A(\gamma; \tau) \equiv (\gamma/2)(3\tau^4 + \gamma^2\tau^2/2 - \gamma^4/16 + \lambda_0),$$

therefore, for $\lambda_0 > 0$, problem I (problem II) for equation (9) is uniquely solvable for

$$\gamma \in (0, 2\sqrt[4]{\lambda_0}) \quad (\gamma \in (-2\sqrt[4]{\lambda_0}, 0)).$$

For the remaining γ there is normal solvability. For $\lambda_0 \leq 0$, both problems are only normally solvable.

Remark. Let

$$P_s(\partial/\partial t, \partial/\partial x_1, \dots, \partial/\partial x_n)u = h(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad (10)$$

be a homogeneous strictly hyperbolic equation of order s . As is known (see ⁽⁶⁻⁸⁾), for equation (9) in Sobolev-Slobodetskii spaces with weight $e^{-\gamma t}$, for $\gamma > 0$, the Cauchy problem is correctly posed. If, however, equation (10) is considered in Sobolev-Slobodetskii spaces with weight $e^{-\gamma t}$ for $\gamma < 0$, then the problem without initial conditions is the correct one.

Moscow Power Engineering Institute

Received
27 III 1969

REFERENCES

- ¹ J. Lions, *Equations différentielles opérationnelles et problèmes aux limites*, Berlin, 1962.
- ² S. G. Krein, *Linear Differential Equations in Banach Space*, Moscow, 1967.
- ³ Yu. A. Dubinskii, *Mat. Sb.*, **79** (121), 1 (1969).
- ⁴ O. A. Ladyzhenskaya, *The Mixed Problem for Hyperbolic Equations*, Moscow, 1953.
- ⁵ O. A. Ladyzhenskaya, *Mat. Sb.*, **39** (81), 4, 383 (1956).
- ⁶ I. G. Petrovskii, *Mat. Sb.*, **2** (44), 815 (1937).
- ⁷ J. Leray, *Hyperbolic Differential Equations*, Princeton, 1952.
- ⁸ L. Gårding, *The Cauchy Problem for Hyperbolic Equations*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.