

# ON A CERTAIN CONJECTURE OF K. WILLIAMS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## **ON A CERTAIN CONJECTURE OF K. WILLIAMS**

*(Presented by Academician Yu. V. Linnik, May 13, 1968)*

In the paper <sup>(1)</sup> the conjecture was put forward that the pair  $(m, m+1)$  of least consecutive residues modulo a prime  $p$  of any integer polynomial of degree  $d$  satisfies the inequality

$$1 \leq m < C\sqrt{p} \log p,$$

where  $C$  is a constant depending only on  $d$ .

In the present note, in particular, this conjecture is confirmed as a special case of a general result on the least residue of a finite system of polynomials, and the estimate is improved by a factor  $\log p$ .

For simplicity we shall restrict ourselves to a system of two polynomials and shall henceforth use the following notation:  $F(X), G(Y)$  are polynomials of degrees  $d, \delta$  with integer coefficients, considered over the field  $[p]$ , where  $p$  is a growing prime number;  $\Phi(X, Y) = (F(X) - F(Y))/(X - Y)$ ;  $r(m), \rho(m)$  are the numbers of solutions of the congruences  $F(X) \equiv m \pmod{p}$  and  $G(Y) \equiv m \pmod{p}$ , respectively ( $1 \leq m \leq p$ );  $\nu$  is the least common natural residue of the polynomials  $F(X), G(Y)$  modulo  $p$ ;  $\mu$  is the least natural number which is simultaneously a nonresidue of  $F(X)$  and a residue of  $G(Y)$ . Throughout, unless explicitly stated otherwise, it is assumed that the constant in the symbol  $O$  depends only on  $d, \delta$ .

**Theorem 1.** *If the polynomial  $F(X) - G(Y)$  has at least one divisor absolutely irreducible over  $[p]$ , and  $1 \leq d, \delta < p$ , then there exists a constant  $C$ , depending only on  $d, \delta$ , such that for all sufficiently large  $p$  the number  $\nu$  exists and*

$$1 \leq \nu < C\sqrt{p}.$$

**Remark.** The condition of the theorem will certainly be fulfilled for all sufficiently large  $p$  if one assumes that  $F(X) - G(Y)$  has at least one divisor absolutely irreducible over the field of rational numbers.

**Corollary.** For all polynomials of degree  $d$  ( $1 \leq d < p$ ), the pair  $(m, m+1)$  of least consecutive residues has the estimate

$$m = O(\sqrt{p}).$$

This is obtained by taking  $G(Y) = F(Y) - 1$ , since the polynomial  $F(X) - F(Y) + 1$  is absolutely irreducible.

**Theorem 2.** Suppose that the following conditions are satisfied:

- 1)  $2 \leq d < p$ ,  $1 \leq \delta < p$ .
- 2) The polynomial  $F(X) - G(Y)$  has exactly one divisor absolutely irreducible over  $[p]$ .
- 3) The curve  $\Phi(X, Y) = 0$ ,  $F(X) - G(Z) = 0$  has at least one absolutely irreducible component defined over  $[p]$ .

Then there exists a constant  $C$ , depending only on  $d, \delta$ , such that for all sufficiently large  $p$  the number  $\mu$  exists and

$$1 \leq \mu < C\sqrt{p}.$$

Let us note that, in a certain sense, the conditions of Theorems 1 and 2 are necessary. For example, if one sets  $F(X) = X^2$ ,  $G(Y) = 2Y^2$ , then for all  $p$  satisfying  $(2/p) = -1$  the polynomial  $X^2 - 2Y^2$  will be irreducible over  $[p]$ , but not absolutely irreducible, and, obviously, the number  $v$  does not exist. If, further, one sets  $F(X) = X^2$ ,  $G(Y) = Y^2$ , then all the conditions of Theorem 2 will be satisfied except condition 2), and the number  $\mu$  does not exist.

Theorem 2 with  $G(Y) = Y$  turns into the known result of Bombieri and Davenport <sup>(2)</sup>, improved in <sup>(3)</sup>.

The proofs are based on the following lemmas.

**Lemma 1.** Let  $u$  be an integer satisfying  $1 \leq u \leq (p-1)/2$ ; let  $k(m)$  be the number of solutions of the congruence  $x + y \equiv m \pmod{p}$ , where  $1 \leq x \leq u$ ,  $1 \leq y \leq u$ . Then for every complex-valued function  $\lambda(m)$  we have:

$$\sum_{m=1}^p \lambda(m)k(m) = \frac{u^2}{p} \sum_{m=1}^p \lambda(m) + O(uR),$$

where the constant in the symbol  $O$  is absolute and

$$R = \max_{1 \leq t \leq p-1} \left| \sum_{m=1}^p \lambda(m) \exp \left( \frac{2\pi i}{p} tm \right) \right|.$$

**Proof** with insignificant changes in notation is contained in <sup>(3)</sup>.

**Lemma 2.** If the polynomial  $F(X) - G(Y)$  has  $e$  distinct factors, absolutely irreducible over  $[p]$ , and  $1 \leq d, \delta < p$ , then

$$\sum_{m=1}^p r(m)\rho(m)k(m) = eu^2 + O(u\sqrt{p}).$$

**Proof.** Apply Lemma 1 with  $\lambda(m) = r(m)\rho(m)$ . The sum

$$\sum_{m=1}^p r(m)\rho(m)$$

is equal to the number of points of the curve  $F(X) - G(Y) = 0$  having coordinates in  $[p]$ . Each component defined over  $[p]$  gives  $p + O(\sqrt{p})$  points by the known results of <sup>(4)</sup>. The number of points common to two or more components, and also the number of points corresponding to components defined over a proper extension of the field  $[p]$ , has estimate  $O(1)$  (see, for example, <sup>(5)</sup>). Consequently,

$$\sum_{m=1}^p r(m)\rho(m) = ep + O(\sqrt{p}).$$

Further, for  $1 \leq t \leq p-1$ ,

$$\sum_{m=1}^p r(m)\rho(m) \exp\left(\frac{2\pi i}{p} tm\right) = \sum \exp\left(\frac{2\pi i}{p} tF(x)\right),$$

where the summation is carried out along the curve  $F(X) - G(Y) = 0$ . By the results of <sup>(5)</sup> on exponential sums along a curve (see also <sup>(2)</sup>) this sum is  $O(\sqrt{p})$ . The assertion now follows from Lemma 1.

**Lemma 3.** Let the following conditions be satisfied:

- 1)  $2 \leq d < p, \quad 1 \leq \delta < p$ .
- 2) The polynomial  $F(X) - G(Y)$  has  $e$  distinct factors, absolutely irreducible over  $[p]$ .
- 3) The curve  $\Phi(X, Y) = 0, F(X) - G(Z) = 0$  has  $e'$  distinct components, absolutely irreducible over  $[p]$ .

Then

$$\sum_{m=1}^p r^2(m)\rho(m)k(m) = (e + e')u^2 + O(u\sqrt{p}).$$

**Proof.** Setting  $\lambda(m) = r^2(m)\rho(m)$  and arguing in the same way as above, we arrive at summation along the curve  $F(X) - F(Y) =$

$= 0, F(X) - G(Z) = 0$ , which is representable as a union of the curves  $X - Y = 0, F(X) - G(Z) = 0, \Phi(X, Y) = 0, F(X) - G(Z) = 0$ . These curves give, respectively,  $e(p + O(\sqrt{p}))$  and  $e'(p + O(\sqrt{p}))$  points. The remainder term  $R$  is estimated in the same way as in Lemma 2.

**Lemma 4.** If  $1 \leq \delta < p$ , then

$$\sum_{m=1}^p \rho(m)k(m) = u^2 + O(u\sqrt{p}).$$

This follows from Lemma 2 for  $F(X) = X$ .

**Proof of Theorem 1.** Setting in Lemma 2  $u = C_1\sqrt{p}$ , where  $C_1$  is a suitably chosen constant, and taking into account that  $e \geq 1$  and  $k(m) = 0$  when  $m > 2u$ , we obtain

$$\sum_{m=1}^{2u} r(m)\rho(m)k(m) > 0$$

and, consequently,  $\nu$  exists and has the estimate  $\nu = O(\sqrt{p})$ .

**Proof of Theorem 2.** Setting

$$N_h = \sum_{r(m)=h} \rho(m)k(m)$$

( $0 \leq h \leq d$ ) and applying Lemma 4 and Lemmas 2, 3 for  $e = 1$ ,  $e' \geq 1$ , we obtain (see (2))

$$N_0 = \sum_{\substack{r(m)=0 \\ 1 \leq m \leq 2u}} \rho(m)k(m) > 0,$$

where  $u = O(\sqrt{p})$ .

It would be interesting to obtain an analogous result for the general least non-residue of polynomials  $F$  and  $G$ ; however, the method presented here in this case requires refinement.

The estimates of exponential sums used here can be generalized to sums of a more general form

$$\sum_P e(\varphi(P))\chi(\psi(P)),$$

where  $e, \chi$  are additive and multiplicative characters of the finite field  $k$ ;  $\varphi, \psi$  are functions on an algebraic curve defined over  $k$ , and the summation is over the points  $P$  of the curve rational over  $k$ . Using such estimates, one can refine the result of Theorems 1 and 2 in the case when  $G(Y) = Y^k$ , where  $k \mid p-1$ . The following can be proved.

**Theorem 3.** Let the integer  $k \mid p - 1$ ; let  $F(X)$  be a polynomial of degree  $d$ , where  $2 \leq d < p$ . Suppose that the following conditions are satisfied:

- 1) The polynomial  $\Phi(X, Y)$  has at least one factor absolutely irreducible over  $[p]$ .
- 2)  $F(X) \neq af^l(X)$ , where  $a \in [p]$ ,  $f(X)$  is a polynomial over  $[p]$ ,  $l \mid k$ ,  $l > 1$ .

Then there exists a constant  $C$ , depending only on  $d$ , and  $k$ -th power residues  $\lambda, \mu \pmod{p}$ , satisfying the conditions:

- 1)  $1 \leq \lambda, \mu < Ck\sqrt{p}$ .
- 2) The congruence  $F(X) \equiv \lambda \pmod{p}$  is solvable.
- 3) The congruence  $F(X) \equiv \mu \pmod{p}$  is not solvable.

*Note added in proof.* K. Williams proved his conjecture in paper (6), which became known to the author only after the present article had been submitted for publication. However, K. Williams' s result is obtained as a very special consequence of our results.

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*Note: Figure translations are in progress. See original paper for figures.*

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