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Abstract

Full Text

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MATHEMATICS

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ON THE PROBLEM OF ARITHMETICITY OF DISCRETE GROUPS

(Presented by Academician A. N. Kolmogorov, 3 XII 1968)

I. Definitions. A **discrete subgroup** of a topological group is a subgroup that intersects some neighborhood of the identity only in the identity.

An **algebraic group** is a subgroup of the group $GL(n, \mathbb{C})$ singled out by the conditions that a certain number of polynomials in the matrix entries vanish. If these polynomials can be chosen so that all their coefficients belong to the field of rational numbers $\mathbb{Q}$, then one says that this group is **defined over the field** $\mathbb{Q}$. If k is some ring, then G_k denotes the set of points of the group G defined over k , whose determinant is the identity of the ring k . If $G_R = G \cap GL(n, \mathbb{R})$, then G_Z ($\mathbb{Z}$ is the ring of integers) is called **arithmetic**. Suppose the group G_R is the direct product of a compact group k and some group G_1 , and φ is the projection of G onto G_1 . It is easy to see that $\varphi(G_Z)$ is a discrete subgroup of the group G_1 . Discrete groups obtained by such a construction are also called **arithmetic groups**. Two subgroups are called **commensurable** if their intersection has finite index in each of them. A subgroup commensurable with an arithmetic subgroup is also called **arithmetic**.

An element $u \in G$ is called **unipotent** if all its eigenvalues are equal to 1. A subgroup of a group is called **unipotent** if all its elements are unipotent. The **unipotent radical** of a group $H \subset G$ is the maximal unipotent subgroup $U \subset H \subset G$ that is a normal divisor in H . We shall say that a closed unipotent subgroup is **horospherical** if it coincides with the unipotent radical of its normalizer. A closed subgroup P of the group G is called **parabolic** if the quotient space G/P is compact. It is not difficult to verify that a unipotent subgroup is horospherical if and only if its normalizer is a parabolic subgroup.

II. If G is a semisimple group and Γ is its arithmetic subgroup, then it is known that the volume of the quotient space G/Γ is finite. A. Selberg conjectured that, for almost all semisimple real Lie groups, all their discrete subgroups with quotient space of finite volume are arithmetic.

E. B. Vinberg and V. S. Makarov constructed a series of examples showing that this conjecture, in the case when the group G is the group of motions of Lobachevsky space (in dimensions 3, 4, and 5), is false.

Let us divide all discrete subgroups with quotient space of finite volume into two classes: 1) the quotient space G/Γ is compact; 2) the quotient space G/Γ is noncompact.

In Section III a plan of proof of the following theorem will be outlined:

Main theorem. *Let G be a semisimple real Lie group without compact factors, $\text{rank } G \geq 2$, Γ its discrete subgroup-*

* $\text{rank } G$ is the dimension of a maximal split torus. A split torus is a subgroup consisting of semisimple elements all of whose eigenvalues are real.

is finite, G/Γ is noncompact and the pair (G, Γ) is irreducible*. Then there exist a group G_Q and a parabolic subgroup $P \subset G$ such that $G_R = G$, $\Gamma \cap G_Q$ is commensurable with Γ , and $\Gamma \cap P$ and $G_Z \cap P$ are commensurable.

III. Let G be a connected Lie group. Then G/R , where R is the maximal solvable divisor in G , is a semisimple algebraic group.

Denote by f the natural mapping $G \rightarrow G/R$.

Theorem 1. *If Γ is a discrete subgroup in G and $f(\Gamma)$ is dense in the Zariski topology in G/R , then $f(\Gamma)$ is discrete.*

The proof of Theorem 1 is in many respects analogous to the proof of Bieberbach's classical theorem on discrete subgroups of the group of motions of n -dimensional Euclidean space.

Introduce on G a right-invariant measure which naturally induces on G/Γ a certain measure μ .

Corollary. *If*

$$\int_{G/\Gamma} d\mu < \infty,$$

then $f(\Gamma)$ is discrete.

Let T^t be a one-parameter group consisting of unipotent linear transformations of Euclidean space $\mathbb{R}^n$. The group T^t induces a group of transformations $\tilde{T}^t$ in the space of lattices of full dimension in $\mathbb{R}^{n**}$. Consider in $\mathbb{R}^n$ a lattice $\mathbb{Z}^n$ of full dimension such that the unit neighborhood of 0 in $\mathbb{R}^n$ contains no nonzero point of the lattice $\mathbb{Z}^n$.

Theorem 2. *There exists a $\delta > 0$ such that, for any $t_0 > 0$, there is a $t > t_0$ for which the intersection of the δ -neighborhood of 0 in $\mathbb{R}^n$ with $\tilde{T}^t \mathbb{Z}^n$ consists only of 0.*

By Mahler's compactness criterion, Theorem 2 can be reformulated as follows:

Theorem 2'. *If the trajectory of a point $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ under the action of a one-parameter unipotent group $U \in SL(n, \mathbb{R})$ is closed, then it is compact.*

Corollary. *Let H^{***} be a semisimple algebraic subgroup of the group $SL(n, \mathbb{R})$, and let x be some point in the space $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Then, if the trajectory Hx of the point x under the action of the group H is closed, it has finite volume (with respect to the natural measure).*

In the proof of the corollary one uses the theorem that every affine algebraic variety in which the set of rational points is dense in the Zariski topology is defined over $\mathbb{Q}$.

In what follows in this section it will be assumed that G is a semisimple algebraic real Lie group without compact factors, Γ is its discrete subgroup,

$$\int_{G/\Gamma} d\mu < \infty,$$

G/Γ is noncompact, and the pair (G, Γ) is irreducible.

Consider a maximal unipotent subgroup U_Γ in the discrete group Γ , and take a minimal connected unipotent subgroup U_G containing U_Γ .

Using Theorems 1 and 2 and the methods of [5], Theorems 3 and 4 are proved.

Theorem 3. U_G is a horospherical subgroup of the group G .

Theorem 4. *There exists a parabolic subgroup $P \subset G$ such that $P \cap \Gamma$ is an arithmetic subgroup in P .*

Suppose now that $\text{rank } G \geq 2$.

* We shall call the pair (G, Γ) irreducible if, for any decomposition of G into a nontrivial direct product of Lie groups $G = G_1 \times G_2$, the product $(G_1 \cap \Gamma) \times (G_2 \cap \Gamma)$ is incommensurable with Γ .

** The topology in the space of lattices is natural.

*** In fact, it suffices to assume that H is an algebraic group having no nontrivial multiplicative characters.

Let U_P denote the unipotent radical of the group P , and let g_P be the natural mapping $P \rightarrow P/U_P$. Consider in G two parabolic subgroups P_1 and P_2 satisfying the conclusion of Theorem 4. Then the following holds.

Theorem 5. $g_{P_1}(P_1 \cap P_2 \cap \Gamma)$ is commensurable with $g_{P_1}(P_1 \cap \Gamma)$.

One can prove the following algebraic theorem.

Theorem 6. Let P be a parabolic subgroup of the group G , defined over $\mathbb{Q}$. Then there exists a unique group G_Q such that $G_R = G$ and $G_Q \cap P = P_Q$.

With the aid of Theorems 4, 5, and 6 the main theorem is proved.

IV. A fundamental domain of a discrete group Γ is a set which intersects each set of the form Γg (the trajectory of the element g under the action of the group Γ) in exactly one point. If O is a fundamental domain, then γO , where $\gamma \in \Gamma$, is also a fundamental domain. The number of sides of a fundamental domain is the number of those elements $\gamma \in \Gamma$ for which γO borders on O (i.e., the closure of γO intersects the closure of O).

From Theorem 4 it follows that

Theorem 7. If $\int_{G/\Gamma} d\mu < \infty$, then for Γ there exists a fundamental domain with a finite number of sides.

Theorem 7 admits the following refinement:

Theorem 7'. There exists a constant c , depending only on the group G , such that, if $\int_{G/\Gamma} d\mu < V$, then there exists a fundamental domain with a number of sides less than cV .

Using the methods of the works ⁽¹⁻⁴⁾ and Theorems 4-6 and the main theorem, one proves

Theorem 8. Let $\text{rank } G \geq 2$, $\Gamma \subset G$, $\int_{G/\Gamma} d\mu < \infty$, the pair (G, Γ) be irreducible, and let T be a finite-dimensional representation of the group G . Then any finite-dimensional representation T' of the group Γ , close to the restriction of T to Γ , is equivalent to it.

Theorem 8 proves Selberg's conjecture on the rigidity of finite-dimensional representations of discrete groups.

The following is related to Theorem 7.

Theorem 9. For a given group G there exists only a finite number of nondiffeomorphic manifolds of the form G/Γ whose volume is less than some previously given constant V .

From Mostow's theorem ⁽⁶⁾ and Theorem 9 one derives

Corollary. If $n \geq 3$, then there exists only a finite number of nonisometric manifolds of constant negative curvature whose volume is less than some previously given constant V .

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