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1969

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Abstract

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UDC 519.45

MATHEMATICS

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TWO PROPOSITIONS ON BAER GROUPS

(Presented by Academician P. S. Novikov on 14 XI 1968)

Let there be given a certain operator U , assigning to every class of groups $\mathfrak{X}$ a definite class $U\mathfrak{X}$. The powers of the operator U are defined as follows: for any class of groups $\mathfrak{X}$ we put $U^0\mathfrak{X} = \mathfrak{X}$ and suppose that for all transfinite ordinal numbers β less than α , the operators U^β have already been defined. If α is a limit ordinal, put

$$U^\alpha\mathfrak{X} = \bigcup_{\beta < \alpha} U^\beta\mathfrak{X},$$

whereas if $\alpha = \gamma + 1$, then $U^\alpha\mathfrak{X} = U(U^\gamma\mathfrak{X})$. In addition, let

$$\overline{U}\mathfrak{X} = \bigcup_{\alpha} U^\alpha\mathfrak{X}.$$

The operator $\overline{U}$ is called the closure of the operator U .

For an arbitrary class of groups $\mathfrak{X}$, denote by $R\mathfrak{X}$ the class of groups generated by their normal divisors from $\mathfrak{X}$, by $K\mathfrak{X}$ the class of groups covered by their normal divisors from $\mathfrak{X}$, and by $F\mathfrak{X}$ the class of groups in which every finite set of elements is contained in a normal divisor from the class $\mathfrak{X}$.

Let $\mathfrak{A}$ denote the class of abelian groups, $\mathfrak{N}_f$ the class of finitely generated nilpotent groups, $\mathfrak{N}$ the class of nilpotent groups, $\mathfrak{B}$ the class of Baer groups, and F the class of finite groups. Groups from the class $R\mathfrak{N}$ are called fitting groups.

1. As is known, $\overline{R\mathfrak{A}} = \overline{R\mathfrak{N}} = \overline{F\mathfrak{N}_f} = \mathfrak{B}$. Recently R. S. Dark in ⁽¹⁾ constructed an example of a primary and, consequently, non-fitting Baer group, thereby showing that the classes $R\mathfrak{N}$ and $\overline{R\mathfrak{N}}$ are distinct. More complete information is given by the following theorem.

Theorem 1. *For every integer $n \geq 1$ there exists a Baer group which is not generated by its n -generated nilpotent subgroups.*

Proof. Fix a prime number $p \geq n + 1$. Let H be the discrete direct product of a countable set of cyclic groups of order p . Construct a sequence of groups by putting

$$G_0 = H, \quad G_i = \{a_i\} \text{ wr } G_{i-1}, \quad \text{where } a_i^p = 1, \quad i = 1, 2, \dots$$

By successive application of the following known lemma it is proved that each G_i is a Baer group.

Lemma 1. *If a normal divisor B_0 of a group B is an abelian p -group of finite exponent, and B/B_0 is a Baer p -group, then B is also a Baer p -group.*

In the wreath product $A \wr B$ of two groups A and B , the copy of the group A corresponding to an element $b \in B$ is denoted by A_b , and the element of the group A_b corresponding to an element $a \in A$ is denoted by a_b .

Lemma 2. *Let $G = A \wr B$, where B is a p -group. Then for any elements $g, f \in B$, $g \neq 1$, $a \in A$, with $m < p$,*

$$[a_f, g(m)] = \prod_{i=0}^m a_{fg^i}^{r_i} \neq 1, \quad \text{where } r_i = (-1)^{m-i} \binom{m}{i}.$$

In (2), p. 394, the following assertion is given.

Lemma 3. *Let $G = A \wr B$, $g \in A^B$, $g \neq 1$, and let $\mathfrak{Z}(g)$ be the centralizer of the element g . Then $\mathfrak{Z}(g) \cap B$ is a finite group.*

We shall show that the group $G_{n+1} = \{a_{n+1}\}^{G_n} \lambda \dots \lambda \{a_1\}^{G_0} \lambda H$ is not generated by its n -generated nilpotent subgroups. Suppose the contrary. Consider the normal divisors

$$H_m = \{a_{n+1}\}^{G_n} \lambda \dots \lambda \{a_{n-m+2}\}^{G_{n-m+1}}, \quad 1 \leq m \leq n+1,$$

and also let $H_0 = E$. In G_{n+1} there is a normal series

$$E \subset F_1 \subset F_2 \subset \dots \subset F_{n-1} \subset F_n \subset G,$$

where F_1 is a nilpotent subgroup not lying in H_{n+1} . Take an element $g \in F_1 \setminus H_{n+1}$, and let $g_i = [b_i, g(n)]$, $i = 1, 2, \dots, n+1$, where b_i is an arbitrary representative of the element a_i in the group $\{a_i\}^{G_{i-1}}$. From Lemma 2 it follows that $g_i \in H_{n-i+2} \setminus H_{n-i+1}$, $1 \leq i \leq n+1$. Moreover, since F_1 is of class n , all $g_i \in F_1$.

Construct the subgroups $M_1 = \{h^{-1}g_1h \mid h \in H\}$, $M_2 = \{h^{-1}g_2h \mid h \in M_1\}$, $\dots$, $\dots, M_n = \{h^{-1}g_nh \mid h \in M_{n-1}\}$. By construction, $M_i \subset H_{n-i+2}$, $1 \leq i \leq n$. The subgroup M_1 lies in F_n , since it is generated by elements conjugate to $g_1 \in F_1 \subset F_n$. Next, M_2 lies in F_{n-1} , since it is generated by elements conjugate to $g_2 \in F_1 \subset F_{n-1}$ by means of elements from $M_1 \subset F_n$. Continuing, we obtain that M_n lies in F_1 .

If C is a subgroup of the semidirect product $A \lambda B$, then the image of C under the homomorphism $\mu : A \lambda B \rightarrow B$ will be called the projection of C onto B .

Consider the projection of M_1 onto $\{a_1\}^{G_0}$. Let $g_1 = h_2h_1$, $h_1 \in \{a_1\}^{G_0}$, $h_2 \in H_n$. Since $g_1 \in H_{n+1} \setminus H_n$, we have $h_1 \neq 1$. All elements $h^{-1}h_1h$,

where $h \in H$, lie in the projection of M_1 onto $\{a_1\}^{G_0}$. If there are only finitely many such elements, then among them there is an element commuting with an infinite set of elements of H . But this is impossible by Lemma 3, and therefore the projection of M_1 onto $\{a_1\}^{G_0}$ is an infinite group. It is proved analogously that if the projection of M_i onto $\{a_i\}^{G_{i-1}}$ is infinite, then the projection of M_{i+1} onto $\{a_{i+1}\}^{G_i}$ is also infinite. Hence it follows that the projection of M_n onto $\{a_n\}^{G_{n-1}}$ is an infinite group.

The nilpotent subgroup F_1 , as was shown, intersects nontrivially the normal divisor $H_1 = \{a_{n+1}\}^{G_n}$. Then the center of the group F_1 also intersects H_1 nontrivially. Let $f \in Z(F_1) \cap H_1$, $f \neq 1$. The element f commutes with $M_n \subset F_1$. In this case f also commutes with the projection of M_n onto $\{a_n\}^{G_{n-1}}$. But, since this projection is infinite, this is impossible by Lemma 3. The contradiction obtained proves the theorem.

Consequence. In the sequence

$$\mathfrak{N} \subset R\mathfrak{N} \subset R^2\mathfrak{N} \subset \dots \subset R^m\mathfrak{N} \subset R^{m+1}\mathfrak{N} \subset \dots \subset R^\omega\mathfrak{N} \subset R^{\omega+1}\mathfrak{N}$$

all inclusions are proper.

Indeed, the discrete direct product of the groups constructed in Theorem 1 over all n gives a Baer group not lying in $R^\omega\mathfrak{N}$. Therefore neither the equality $R^\omega\mathfrak{N} = R^{\omega+1}\mathfrak{N}$, nor $R^m\mathfrak{N} = R^{m+1}\mathfrak{N}$ for any natural m , can hold.

On the other hand, it is known ⁽³⁾ that for any class of groups $\mathfrak{X}$

$$R^{\omega+1}\mathfrak{X} = R^{\omega+2}\mathfrak{X} = \overline{R\mathfrak{X}}$$

and, in particular, $R^{\omega+1}\mathfrak{N} = \overline{R\mathfrak{N}}$.

2. The operator $\overline{K}$ was first considered by P. G. Kontorovich, who, in particular, considered the class of groups $\overline{K\mathfrak{A}}$, which he called groups with a category ⁽⁴⁾. The following theorem shows that in a number of cases the operator $\overline{K}$ coincides in its action with the operator of radical closure $\overline{R}$.

Theorem 2. *Let $\mathfrak{X}$ be a class of groups satisfying the following requirements:*

- 1) *all groups of the class $\mathfrak{X}$ are nonzero;*
- 2) *in any group the product of two normal divisors from the class $\mathfrak{X}$ belongs to the class $\overline{F\mathfrak{X}}$;*
- 3) *the class $\mathfrak{X}$ is closed with respect to homomorphisms and the taking of subgroups. Then the equality $F\mathfrak{X} = \overline{K\mathfrak{X}} = \overline{R\mathfrak{X}}$ holds.*

Proof. For any class of groups $\mathfrak{X}$ there are inclusions $F\mathfrak{X} \subset K\mathfrak{X} \subset R\mathfrak{X}$, whence it follows that $F\mathfrak{X} \subset \overline{K\mathfrak{X}} \subset \overline{R\mathfrak{X}}$. The class $\overline{R\mathfrak{X}}$ is the minimal radical class over $\mathfrak{X}$. We shall prove that if $\mathfrak{X}$ satisfies the conditions of the theorem, then $\overline{F\mathfrak{X}}$ is a radical class, whence the inclusion $\overline{R\mathfrak{X}} \subset \overline{F\mathfrak{X}}$, proving the theorem, will follow.

Since in any group the union of an increasing sequence of normal divisors from $\bar{F}\mathfrak{X}$, obviously, also belongs to $\bar{F}\mathfrak{X}$, it is enough to prove the following lemma.

Lemma 4. If a class of groups $\mathfrak{X}$ satisfies the conditions of the theorem, then in any group the product of two normal divisors from $\bar{F}\mathfrak{X}$ also belongs to the class $\bar{F}\mathfrak{X}$.

Proof. It is enough to show that if some group G is generated by its normal divisors A_1 and A_2 , belonging to $\bar{F}\mathfrak{X}$, then $G \in \bar{F}\mathfrak{X}$. For some ordinal number α , $A_1 \in F^\alpha\mathfrak{X}$ and $A_2 \in F^\alpha\mathfrak{X}$. If $\alpha = 0$, then $G \in F\mathfrak{X}$ by condition 2) of the theorem. Suppose that $\alpha > 0$ and that for all β less than α it has already been proved that in any group the product of two normal divisors from $F^\beta\mathfrak{X}$ belongs to the class $\bar{F}\mathfrak{X}$.

We note that, according to conditions 2) and 3) of the theorem, the class of groups $\bar{F}\mathfrak{X}$ is closed with respect to homomorphisms, the taking of subgroups, and finite direct products.

Consider in the group G all possible normal series

$$G = G_1 \supset G_2 \supset \dots \supset G_n \supset G_{n+1} \supset \dots,$$

where G_n is a normal divisor in G_{n-1} , generated by some finite set of elements M_{n-1} . For the group G to belong to the class $\bar{F}\mathfrak{X}$ it is necessary that in each such series, for some natural n , $G_n \in \mathfrak{X}$, and it is sufficient that $G_n \in \bar{F}\mathfrak{X}$.

Writing each $g \in M_n$ in the form $g = a_1 a_2$, $a_1 \in A_1$, $a_2 \in A_2$, we construct finite sets $M'_n \subset A_1$ and $M''_n \subset A_2$, $M_n \subset M'_n M''_n$, and let

$$G = H_1 \supset H_2 \supset \dots \supset H_n \supset H_{n+1} \supset \dots$$

be such a normal series in G that H_n is a normal divisor in H_{n-1} , generated by the set $M'_n \cup M''_n$ and by the subgroup $A_1 \cap A_2 = A$. Since the group G/A is the direct product of its normal divisors A_1/A and A_2/A , by the remark made above it belongs to $\bar{F}\mathfrak{X}$. Consequently, there exists such an n that $H_n/A \in \mathfrak{X}$. We shall prove that then $G_{n+1} \in \bar{F}\mathfrak{X}$, and thereby it will be shown that $G \in \bar{F}\mathfrak{X}$. Consider the normal divisors S , S_1 , and S_2 of the group H_n , generated respectively by the sets $M'_n \cup M''_n$, M'_n , and M''_n . Since $G_n \subset H_n$, we have $G_{n+1} \subset S$. Let X be a finite set of elements whose image in G/A generates the subgroup H_n/A . Denote by T the subgroup of the group G generated by the set $X \cup M'_n \cup M''_n$, by T_1 the normal divisor of the group T generated by the set M'_n , and by T_2 the normal divisor of T generated by M''_n . In view of condition 1) of the theorem, all groups of the class $F\mathfrak{X}$ are locally Noetherian. Therefore the group G , which is generated by its normal divisors A_1 and A_2 belonging to $\bar{F}\mathfrak{X}$, is also locally Noetherian. Consequently, the group T is Noetherian, and its subgroups T_1 and T_2 have finite systems of generators Y_1 and Y_2 . It is clear that $Y_1 \subset A_1$ and $Y_2 \subset A_2$. Let B_1 and B_2 be the normal divisors in A_1 and A_2 , respectively, generated by the sets Y_1 and Y_2 . Since Y_1 and Y_2 are finite sets, $B_1 \in F^\beta\mathfrak{X}$ and $B_2 \in F^\beta\mathfrak{X}$ for some $\beta < \alpha$.

Since T_1 and A_1 are invariant with respect to X , the subgroup B_1 is also invariant with respect to X ; and since H_n is contained in the subgroup generated by the set X and A_1 , B_1 is invariant with respect to H_n . Similarly, the subgroup B_2 is invariant with respect to H_n . Moreover, we have $M'_n \subset T_1 \subset B_1$, $M''_n \subset T_2 \subset B_2$, whence $S_1 \subset B_1$ and $S_2 \subset B_2$. Consequently, $S_1 \in F^{\beta}\mathfrak{X}$ and $S_2 \in F^{\beta}\mathfrak{X}$. Therefore, by the induction hypothesis,

$$S = S_1 S_2 \in \bar{F}\mathfrak{X}.$$

Then also $G_{n+1} \in \bar{F}\mathfrak{X}$. The lemma, and together with it the theorem, are proved.

Corollary 1. Every Baer group is a group with categories.

Indeed, the class $\mathfrak{A}_f$ satisfies the condition of Theorem 2 and, consequently,

$$K\mathfrak{A} = K\mathfrak{A}_f = R\mathfrak{A}_f = \mathfrak{B}.$$

Corollary 2. Every periodic Baer group is contained in the class $F\mathfrak{F}$ of generalized locally normal groups.

The class of groups $F\mathfrak{F}$ was introduced in the paper ⁽⁵⁾.

I express my deep gratitude to Prof. B. I. Plotkin, under whose supervision the present note was written. I also thank V. G. Vil' yatser for a useful discussion of the second part of the note.

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Received
24 X 1968

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Note: Figure translations are in progress. See original paper for figures.

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