

ON ONE SOLUTION OF THE ONE-DIMENSIONAL DIFFUSION EQUATION IN A BOUNDED DOMAIN

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1969

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Abstract

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UDC 532.72

HYDROMECHANICS

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ON ONE SOLUTION OF THE ONE-DIMENSIONAL DIFFUSION EQUATION IN A BOUNDED DOMAIN

(Presented by Academician L. I. Sedov, May 17, 1968)

A qualitative study of self-oscillations arising under certain conditions in oil production or in some electrolytic systems was given in papers ⁽¹⁻³⁾. In papers ⁽³⁻¹⁰⁾, periodic solutions of the one-dimensional diffusion equation in a semi-infinite domain of variation of the spatial variable x , connected with this problem, were obtained. Below a periodic solution is found for a certain diffusion problem in a bounded domain.

As is known ⁽¹¹⁾, the diffusion equation

$$\partial c / \partial t = D \partial^2 c / \partial x^2 \quad (1)$$

has periodic solutions of the form

$$Ae^{\pm\sqrt{\omega/2D}x} \cos(\mp\sqrt{\omega/2D}x + \omega t), \quad Ae^{\pm\sqrt{\omega/2D}x} \sin(\mp\sqrt{\omega/2D}x + \omega t). \quad (2)$$

For the semi-infinite interval $0 < x < \infty$, a periodic solution of equation (1) was obtained in paper ⁽⁴⁾.

Let us now consider the following problem: to find a periodic solution of equation (1) for the finite interval $0 < x < l$, satisfying the boundary condition

$$c(l, t) = c^0. \quad (3)$$

We shall seek the function $c(x, t)$ in the form

$$c(x, t) = c_0 + (c^0 - c_0)x/l + u(z, \tau), \quad (4)$$

where $u(z, \tau)$ is a periodic solution of the equation

$$\partial u / \partial \tau = \partial^2 u / \partial z^2 \quad (\tau = Dt/l^2, \quad z = x/l - 1) \quad (5)$$

with period T , satisfying the conditions

$$u(0, \tau) = 0, \quad \int_0^T u(z, \tau) d\tau = 0. \quad (6)$$

Using formulas (2), (5), and (6), it is easy to verify that the desired solution can be represented in the form

$$u(z, \tau) = \sum_{k=1}^{\infty} u_k(z, \tau), \quad u_k(z, \tau) = A_k \{e^{\rho_k z} \cos(\omega_k \tau + \rho_k z) - \quad (7)$$

$$-e^{-\rho_k z} \cos(\omega_k \tau - \rho_k z)\} + B_k \{e^{\rho_k z} \sin(\omega_k \tau + \rho_k z) - e^{-\rho_k z} \sin(\omega_k \tau - \rho_k z)\},$$

where

$$\rho_k = \sqrt{\pi k/T}, \quad \omega_k = 2\pi k/T, \quad (8)$$

and A_k and B_k denote constants which must be determined from the boundary condition.

We shall now seek a periodic solution of equation (1) for the finite interval $0 < x < l$ with the boundary condition

$$\partial c(0, t) / \partial x = \Omega[c(0, t)], \quad (9)$$

where Ω is an S -shaped function ⁽²⁾, at $x = 0$, and with condition (3) at $x = l$.

Using formulas (4) and (5), we reduce this problem to finding a solution of the diffusion equation (5) in the form (7), where the constants A_k and B_k are found from the boundary condition

$$\partial u(-1, \tau) / \partial z = F[u(-1, \tau)] \quad (F(u) = l\Omega(c_0 + u) - c^0 + c_0). \quad (10)$$

Introducing the notation

$$\begin{aligned} \alpha_k(z) &= 2(A_k \cos \rho_k z \operatorname{sh} \rho_k z + B_k \sin \rho_k z \operatorname{ch} \rho_k z), \\ \beta_k(z) &= 2(-A_k \sin \rho_k z \operatorname{ch} \rho_k z + B_k \cos \rho_k z \operatorname{sh} \rho_k z), \end{aligned} \quad (11)$$

we write equation (10) in the following form:

$$\begin{aligned} & \sum_{k=1}^{\infty} [\alpha'_k(1) \cos \omega_k \tau + \beta'_k(1) \sin \omega_k \tau] = \\ & = F \left\{ - \sum_{k=1}^{\infty} [\alpha_k(1) \cos \omega_k \tau + \beta_k(1) \sin \omega_k \tau] \right\}. \end{aligned} \quad (12)$$

Equation (12), taking into account the notation (11), gives an infinite system of nonlinear equations for finding the constants A_k and B_k ($k = 1, 2, \dots$).

If $F[u(-1, \tau)] = \chi(\tau)$, where $\chi(\tau)$ is a known periodic function of τ with period T , satisfying, by virtue of (6), the condition

$$\int_0^T \chi(\tau) d\tau = 0, \quad (13)$$

then, expanding $\chi(\tau)$ in a Fourier series, one can find the coefficients A_k and B_k from (12) and (11).

Let us consider in more detail the case where $F(u)$ is a piecewise-constant function

$$F(u) = Q_1 \quad \text{for } 0 < \tau < T_1; \quad F(u) = Q_2 \quad \text{for } T_1 < \tau < T. \quad (14)$$

It follows from (6) that in this case the constants Q_1, Q_2, T_1 , and T must be related by

$$Q_1 T_1 + Q_2 (T - T_1) = 0. \quad (15)$$

Formulas (12), (11), and (13) give the following expressions for the coefficients A_k and B_k :

$$\begin{aligned} A_k &= \frac{(Q_1 - Q_2)}{\pi k \rho_k} \sin \frac{\pi k T_1}{T} \left[(\mu_k - \nu_k) \cos \frac{\pi k T_1}{T} - (\mu_k + \nu_k) \sin \frac{\pi k T_1}{T} \right], \\ B_k &= \frac{(Q_1 - Q_2)}{\pi k \rho_k} \sin \frac{\pi k T_1}{T} \left[(\mu_k + \nu_k) \cos \frac{\pi k T_1}{T} + (\mu_k - \nu_k) \sin \frac{\pi k T_1}{T} \right], \end{aligned} \quad (16)$$

$$\mu_k = \operatorname{ch} \rho_k \cos \rho_k / (\operatorname{ch} 2\rho_k + \cos 2\rho_k), \quad \nu_k = \operatorname{ch} \rho_k \sin \rho_k / (\operatorname{ch} 2\rho_k + \cos 2\rho_k).$$

We now give the solution of the desired problem in another form. Suppose first that the boundary condition at $z = -1$ has the form

$$\partial u(-1, \tau) / \partial z = \chi(\tau), \quad (17)$$

where the periodic function $\chi(\tau)$, for which equality (13) is satisfied, is known. Consider equation (1). The source function $G(x, t, \xi, \tau)$ for the interval $(0 < x < l)$, satisfying the conditions $\partial G(0, t, \xi, \tau) / \partial x = G(l, t, \xi, \tau) = 0$, has the form ^(11,3):

$$G(x, t, \xi, \tau) = \frac{1}{2\sqrt{\pi D(t-\tau)}} \sum_{k=-\infty}^{\infty} (-1)^k \times \\ \times \left[\exp \left(-\frac{(x-\xi-2kl)^2}{4D(t-\tau)} \right) + \exp \left(-\frac{(x+\xi-2kl)^2}{4D(t-\tau)} \right) \right].$$

It can also be represented in the following way:

$$G(x, t, \xi, \tau) = \frac{2}{l} \sum_{k=0}^{\infty} \exp \left[-\frac{(2k+1)^2 \pi D}{4l^2} (t-\tau) \right] \times \\ \times \cos \frac{(2k+1)\pi \xi}{2l} \cos \frac{(2k+1)\pi x}{2l}.$$

Using the source function, we write the solution of equation (5) for $x = 0$ with boundary conditions (6) and (17)

$$u(0, \tau) = - \int_{-\infty}^{\tau} K(\tau - \sigma) \chi(\sigma) d\sigma \quad \left(K(\sigma) = \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \sigma} \right] / \sqrt{\pi \sigma} \right). \quad (18)$$

Let now, analogously to (6),

$$\chi(\tau) = \psi_i(\tau) \quad \text{for } \alpha_i + kT < \tau < \beta_i + kT \\ (i = 1, 2; \quad k = 0, \pm 1, \pm 2, \dots) \quad (19)$$

$$\alpha_1 = pT/2, \quad \alpha_2 = \beta_1 = T - pT/2, \quad \beta_2 = T + pT/2, \quad 0 < p < 1.$$

It is easy to see that

$$u_i(0, \tau) = - \int_{\alpha_i}^{\tau} K(\tau - \sigma) \psi_i(\sigma) d\sigma + S_i(\tau) \quad (\alpha_i \leq \tau \leq \beta_i), \quad (20)$$

$$S_i(\tau) = - \sum_{k=0}^{-\infty} \left\{ \int_{\beta_i - T + kT}^{\alpha_i + kT} K(\tau - \sigma) \varphi_i(\sigma) d\sigma + \int_{\alpha_i - T + kT}^{\beta_i - T + kT} K(\tau - \sigma) \psi_i(\sigma) d\sigma \right\}$$

$$(\varphi_1 = \psi_2, \varphi_2 = \psi_1).$$

The uniform convergence of the series $S_i(\tau)$ follows from the fulfillment of condition (13) and from the fact that $K(\sigma)$ is a monotonically decreasing function of its argument.

Consider again the case of a piecewise-constant function $F(u)$ of the form (14). Using the following relation from (18),

$$f(a, b) = \int_a^b K(y) dy = \frac{2}{\sqrt{\pi}} \left\{ \sqrt{b} - \sqrt{a} + 2 \sum_{n=1}^{\infty} (-1)^n \left[n\sqrt{\pi} \left(\Phi\left(\frac{n}{\sqrt{b}}\right) - \Phi\left(\frac{n}{\sqrt{a}}\right) \right) + \sqrt{b}e^{-n^2/b} - \sqrt{a}e^{-n^2/a} \right] \right\}$$

$$\left(\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy \right), \quad (21)$$

we write the solution $u(0, \tau)$ in the following way:

$$u_i(0, \tau) = Q_i f(\tau - \alpha_i, 0) + S_i(\tau) \quad (\alpha_i \leq \tau \leq \beta_i),$$

$$S_i(\tau) = \sum_{k=0}^{-\infty} \{ q_i f(\tau - \beta_i + T - kT, \tau - \alpha_i - kT) + Q_i f(\tau - \alpha_i + T - kT,$$

$$\tau - \beta_i + T - kT) \} \quad (q_1 = Q_2, q_2 = Q_1). \quad (22)$$

Using condition (13), it is easy to see that $du_1(0, \tau)/d\tau > 0$, $du_2(0, \tau)/d\tau < 0$.

Let us return to the more general case, when the periodic function $\chi(\tau)$, defined by formula (19), is given. From formulas (20) we find

$$\frac{du_i(0, \tau)}{d\tau} = -\frac{1}{\sqrt{\pi}} \left\{ \frac{\psi_i(\alpha_i)}{\sqrt{\tau - \alpha_i}} + \int_{\alpha_i}^{\tau} \frac{\psi'_i(\sigma) d\sigma}{\sqrt{\tau - \sigma}} + \right.$$

$$\left. + \int_{\alpha_i}^{\tau} \frac{\psi_i(\sigma)}{(\tau - \sigma)^{5/2}} \sum_{n=1}^{\infty} [(-1)^n e^{-n^2/(\tau - \sigma)} (2n^2 - \tau + \sigma)] d\sigma \right\} + \frac{dS_i}{d\tau},$$

$$\frac{dS_i}{d\tau} = - \sum_{k=0}^{-\infty} \left\{ \int_{\beta_i - T + kT}^{\alpha_i + kT} K'(\tau - \sigma) \varphi_i(\sigma) d\sigma + \int_{\alpha_i - T + kT}^{\beta_i - T + kT} K'(\tau - \sigma) \psi_i(\sigma) d\sigma \right\} \\ (\alpha_i < \tau < \beta_i). \quad (23)$$

It is easy to see that, by virtue of (13), $dS_1/d\tau > 0$. It can be shown that, if $\psi_1(\tau) < 0$, $\psi_2(\tau) > 0$, $(1-p)T < 2$, $pT < 2$, then, in order for the inequalities $du_1(0, \tau)/d\tau > 0$ ($\alpha_1 < \tau < \beta_1$), $du_2(0, \tau)/d\tau < 0$ ($\alpha_2 < \tau < \beta_2$) to hold, it is sufficient that the relations ($i = 1, 2$) hold.

$$(-1)^i \left[\frac{\psi_i(\alpha_i)}{\sqrt{\tau - \alpha_i}} + \int_{\alpha_i}^{\tau} \frac{\psi'_i(\sigma) d\sigma}{\sqrt{\tau - \sigma}} - \int_{\alpha_i}^{\tau} \frac{e^{-1/(\tau - \sigma)} (2 - \tau + \sigma) \psi_i(\sigma) d\sigma}{(\tau - \sigma)^{5/2}} \right] > 0. \quad (24)$$

The function F entering condition (10) may then be regarded as having the form

$$F(u, \partial u / \partial \tau) = \begin{cases} F_1(u) & \text{for } \partial u / \partial \tau > 0, \\ F_2(u) & \text{for } \partial u / \partial \tau < 0, \end{cases} \quad \begin{matrix} F_1 = \psi_1(\tau), \quad u = u_1(0, \tau), \\ F_2 = \psi_2(\tau), \quad u = u_2(0, \tau). \end{matrix}$$

As an example, let us consider the case in which the functions $\psi_i(\tau)$ in formula (19) are prescribed by the dependences

$$\psi_i(\tau) = e_i + d_i(\beta_i - \tau)^{q_i} + f_i(\tau - \alpha_i)^{1/2}; \quad e_1 < 0, \quad d_1 < 0, \quad f_1 > 0, \quad (25)$$

$$\frac{1}{2} < q_i < 1; \quad e_2 > 0, \quad d_2 > 0, \quad f_2 < 0.$$

A periodic solution of equation (1) with conditions (3) and (17), where $\chi(\tau)$ is given by relations (19) and (25), in the semi-infinite domain $0 < x < \infty$ was found in paper (6). The solution of the corresponding problem in the bounded domain $0 < z < 1$ is given by formulas (20), (19), and (25). Moreover, if we introduce the notation

$$I_1(x, q) = \int_x^{\infty} e^{-z} z^{1/2-q} dz, \quad I_2(x, q) = \int_x^{\infty} e^{-z} z^{q-1/2} dz, \\ I_3(x) = \int_x^{\infty} e^{-z} \sqrt{\frac{z}{x} - 1} dz, \quad I_4(x) = \int_x^{\infty} \frac{e^{-z}}{z} \sqrt{\frac{z}{x} - 1} dz, \quad (26)$$

then inequalities (24), sufficient for having $du_1(0, \tau)/d\tau > 0$, $du_2(0, \tau)/d\tau < 0$, take the form ($p_1 = 1 - p$, $p_2 = p$)

$$(-1)^i \left[e_i (1 - 2e^{-1/p_i T}) - d_i \sqrt{p_i T} \left\{ \frac{(p_i T)^{q_i - 1/2}}{2q_i - 1} + 2I_1 \left(\frac{1}{p_i T}, q_i \right) - I_2 \left(\frac{1}{p_i T}, q_i \right) \right\} + f_i \sqrt{p_i T} \left\{ \frac{\pi}{2} - 2I_3 \left(\frac{1}{p_i T} \right) + I_4 \left(\frac{1}{p_i T} \right) \right\} \right] > 0. \quad (27)$$

From (13) and (25) there also follows a relation among the constants e_i, d_i, f_i, q_i, p and T , coinciding with equality (4.4) of article (6):

$$e_2 p T + \frac{d_2}{q_2 + 1} (p T)^{q_2 + 1} + \frac{2}{3} f_2 (p T)^{3/2} + e_1 (1 - p) T + \frac{d_1}{q_1 + 1} [(1 - p) T]^{q_1 + 1} + \frac{2}{3} f_1 [(1 - p) T]^{3/2} = 0. \quad (28)$$

From (26) it follows that $\lim_{x \rightarrow \infty} I_j(x, q_i) = 0$ ($j = 1, 2, 3, 4$) and $\lim_{x \rightarrow \infty} [(2q_i - 1)x^{q_i - 1/2} I_i(x, q_i)] = 0$ ($i = 1, 2$). Therefore, in the limit as $T \rightarrow 0$, inequalities (27) pass into inequalities (4.3) of article (6).

Thus, if there exists a solution of the problem in the semi-infinite domain, then, at least for sufficiently small values of the period T , there also exists a solution of the corresponding problem in the finite domain. (Indeed, $T \rightarrow 0$ ($l \rightarrow \infty$), since from formula (5) it follows that the period T is related to the actual period T_* of the self-oscillation by the dependence $T = DT_*/l^2$.)

Putting in formulas (20) $\psi_1(\sigma) = F_1[u_1(0, \sigma)]$, $\psi_2(\sigma) = F_2[u_2(0, \sigma)]$, we obtain a nonlinear integral equation for finding the function $u(0, \sigma)$ ($\alpha_1 \leq \sigma \leq \beta_2$).

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Received
8 V 1968

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Note: Figure translations are in progress. See original paper for figures.

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