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**Abstract**

**Full Text**

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**MATHEMATICS**

**P. B. GUSYATNIKOV, M. S. NIKOLSKII**

## ON THE OPTIMALITY OF PURSUIT TIME

*(Presented by Academician L. S. Pontryagin, 20 V 1968)*

Let the motion of the vector  $z$  in an  $n$ -dimensional Euclidean space  $R$  be described by the linear vector differential equation

$$\dot{z} = Cz + u - v, \quad (1)$$

where  $C$  is a constant square matrix of order  $n$ ;  $u \in P$  and  $v \in Q$  are control parameters. The parameter  $u$  corresponds to the pursuing object, the parameter  $v$  to the evading object. The sets  $P$  and  $Q$  are convex compact sets. Let a terminal set  $M$  be given in  $R$ . These data describe the linear pursuit problem (1). The pursuit is considered completed when the point  $z$  first falls into  $M$ .

We shall say that the pursuit can be completed from the point  $z_0$  in finite time if there exists a number  $t(z_0)$  such that, for an arbitrary measurable change of the control parameter  $v(t)$ , one can choose such a measurable change of the control parameter  $u(t)$  that the point  $z$  falls into the set  $M$  in a time not exceeding  $t(z_0)$ ; moreover, to find the value of the parameter  $u(t)$  at each instant of time  $t$ , the values  $z(t)$  and  $v(t)$  at the same instant of time  $t$  are used. With respect to the evader, it is assumed that at the given instant  $t$  he knows  $z(t)$  and the control  $u(s)$  for  $s < t$ . It is in the evader's interest to delay, if possible, the termination of the pursuit; for this purpose he can actively use his information about the pursuer.

2. The basic question in the theory of pursuit is to single out those points  $z_0$  from which the pursuit can be completed in finite time. There are a number of works in which sufficient conditions of general form are given for the possibility of completing the pursuit from a given point  $z_0$  (see <sup>(1-5)</sup>), and the guaranteed time  $t(z_0)$  is effectively computed.

Let, from the given point  $z_0$ , completion of the pursuit in finite time be possible. We shall call the number  $\gamma(z_0)$  the optimal pursuit time if it has the following properties. Let an arbitrary  $\delta > 0$  be given; then: 1) the pursuit can be completed in time  $\gamma(z_0)$ ; 2) for an arbitrary measurable change of the control parameter  $u(t)$ , the evader can construct such a measurable change of

the parameter  $v(t)$ , actively using his information, that the pursuit cannot be completed in a time less than  $\gamma(z_0) - \delta$ .

3. In what follows we shall regard the set  $M$  as a linear subspace. Denote by  $L$  the orthogonal complement in  $R$  to the subspace  $M$ , and by  $\pi$  the operator of orthogonal projection onto  $L$ . Suppose that the dimension of the sets  $P, Q, L$  is equal to  $\nu$ . In this formulation, problem (1) was studied by L. S. Pontryagin in the work <sup>(3)</sup>, which generalizes the results of <sup>(2)</sup>.

In <sup>(3)\*</sup> L. S. Pontryagin regards the sets  $\pi e^{rC}P$ ,  $\pi e^{rC}Q$  ( $r \geq 0$ ) as convex bodies of dimension  $\nu$ . He introduces the operation of subtraction of convex sets and considers the difference  $\pi e^{rC}P \dot{-} \pi e^{rC}Q = \hat{w}(r)$ .

\* The results of the work <sup>(3)</sup> are presented by us somewhat differently and in different notation than this is done in the work itself.

It is assumed that  $\hat{w}(r)$  is a convex  $\nu$ -dimensional body. The concept of the integral of  $\hat{w}(r)$  is introduced:

$$W(t) = \int_0^t \hat{w}(r) dr,$$

where  $W(t)$  is a convex  $\nu$ -dimensional body continuously depending on  $t$ . L. S. Pontryagin shows that if  $T(z_0)$  is the first instant  $t$  at which the inclusion  $-\pi e^{tC}z_0 \in W(t)$  holds, then the pursuit can be completed in time  $T(z_0)$ . In (3) it was assumed that the control  $v(s)$  is known for  $t \leq s \leq t + \varepsilon$ , but, as it turned out, there exists a simple method of pursuit based on current information, which ensures completion of the pursuit exactly at the instant  $T(z_0)$ . The subsequent exposition will be devoted to the study of the time  $T(z_0)$  from the point of view of optimality.

4. From the definition of the subtraction operation  $\dot{-}$  (see (3)) it follows that for any  $r > 0$

$$\pi e^{rC}Q + \hat{w}(r) \subset \pi e^{rC}P$$

(on the left stands the algebraic sum of convex sets). If on a set of nonzero measure in  $r$  a strict inclusion holds, then one says that there is incomplete sweeping. In this case, in constructing the time  $T(z_0)$ , not all of the set  $\pi e^{rC}P$  is used; some corners remain unswept (by corners we mean the set of points of the body  $\pi e^{rC}P$  that do not belong to the body  $\pi e^{rC}Q + \hat{w}(r)$ ). These corners in some problems make it possible for the pursuer to force the time  $T(z_0)$ .

In what follows we shall require that the condition of complete sweeping be satisfied:

$$\pi e^{rC}Q + \hat{w}(r) = \pi e^{rC}P,$$

and then, as is not difficult to see, the time  $T(z_0)$  is the time of first absorption (the definition of this time is given in (5)).

Let us introduce the concept of the maximin time. Suppose that the control  $v(t)$  is known to the pursuer in advance from 0 to  $+\infty$ . Since from the given point  $z_0$  the pursuit can be completed in finite time, there will be a control  $u(t)$  such that it drives the point  $z(t)$  to  $M$  in time  $\theta_{u(t),v(t)}$ . Take the exact lower bound of such times over all possible controls  $u(t)$ , and denote it by  $\theta_{v(t)}$ . Then compute

$$\sup_{v(t)} \theta_{v(t)} = \theta(z_0).$$

Obviously,  $\theta(z_0) \leq T(z_0)$ . If equality holds, then the time  $T(z_0)$  is optimal. However, the example

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = \rho u - \sigma v, \quad (2)$$

where  $z_1, z_2, u, v$  are two-dimensional vectors;  $\rho$  and  $\sigma$  are positive constants, with  $\rho - \sigma > 0$ ;  $|u| \leq 1$ ,  $|v| \leq 1$ , and  $M = \{z : z_1 = 0\}$ , shows that  $\theta(z_0) < T(z_0)$  for an unbounded set of points  $z_0$ . It was found that the time  $T(z_0)$  in this example is optimal, so that the maximin time is not sufficient here for completing the pursuit.

Below we shall formulate condition A, under which we shall prove the optimality of the time  $T(z_0)$ . An example is constructed showing that when condition A is not satisfied, the time  $T(z_0)$ , generally speaking, is not optimal. In this example all the conditions of article (2) are satisfied, and therefore  $T(z_0)$  is the upper-layer time (see (1)). Thus, the upper-layer time is not always optimal.

5. **Condition A.** For every vector  $u$  of the set  $P$  there exists a vector  $v$  of the set  $Q$  such that, independently of  $r$  ( $0 < r \leq T(z_0)$ ), the inclusion

$$\pi e^{rC} u - \pi e^{rC} v \in \hat{w}(r)$$

is satisfied.

This condition relates the matrix  $C$  and the sets  $P, Q, M$ . If the sets  $P$  and  $Q$  lie in nonparallel supporting planes, then the operator  $\pi e^{rC}$  acts differently on these planes, and condition A is not always satisfied.

We shall additionally require that there exist a single-valued mapping  $v = v(u)$  of the set  $P$  into  $Q$  such that, for every measurable function  $u(s)$ , the function  $\tilde{v}(s) = v(u(s))$  is measurable and, independently of  $r$  ( $0 < r \leq T(z_0)$ ), the inclusion holds.

$$\pi e^{rC} u(s) - \pi e^{rC} \tilde{v}(s) \in \hat{w}(r). \quad (3)$$

**Theorem.** If condition A is fulfilled, the time  $T(z_0)$  is optimal.

**Proof.** If the controls  $u(s), v(s)$  are given, then the formula holds

$$\pi z(t) = \pi e^{tC} z_0 + \int_0^t \pi e^{(t-s)C} u(s) - \pi e^{(t-s)C} v(s) ds.$$

The pursuit ends at the moment  $t_1$  when the equality  $\pi z(t_1) = 0$  is first fulfilled.

From the definition of the time  $T(z_0)$  it follows that the expression  $\pi e^{tC} z_0 + \int_0^t w(t, s) ds$  (where  $w(t, s)$  is any function measurable in  $s$  with the restriction  $w(t, s) \in \hat{w}(t - s)$ ) can become the zero vector for the first time only when  $t = T(z_0)$ .

The following mode of behavior is proposed for the evader. During the first  $\varepsilon$  seconds he constructs the control  $v(s)$  arbitrarily, and for  $s > \varepsilon$  he constructs, from the function  $u(s - \varepsilon)$ , a function  $\tilde{v}(s - \varepsilon)$  satisfying inclusion (3) of condition A, and sets  $v(s) = \tilde{v}(s - \varepsilon)$ . It is not hard to show that then

$$\pi z(t) = \pi e^{tC} z_0 + \int_0^t \tilde{w}(t, s) ds + O(\varepsilon), \quad (4)$$

where

$$\tilde{w}(t, s) = \pi e^{(t-s)C} u(s) - \pi e^{(t-s)C} \tilde{v}(s) \in \hat{w}(t - s).$$

Let us note that the set  $W(t) = \left\{ \int_0^t w(t, s) ds \right\}$  is a compact set depending continuously on  $t$ . If one takes any  $\delta > 0$ , then from formula (4) it easily follows that, for sufficiently small  $\varepsilon$ , pursuit cannot be completed in the time  $T(z_0) - \delta$  under the indicated behavior of the evader; and since  $\delta$  may be taken arbitrarily small, the time  $T(z_0)$  proves to be optimal.

6. We shall give two important cases in which condition A is fulfilled.

**Case I.** The sets  $P$  and  $Q$  lie in parallel planes and the equality holds

$$Q + D = P, \quad (5)$$

where  $D$  is a convex  $\nu$ -dimensional compact set.

**Remark.** From equality (5) follows the condition of full sweeping, and  $\hat{w}(r) = \pi e^{rC} D$ . For condition (3) to hold, it is sufficient to require, in addition to  $\dim P = \dim Q = \dim L$ , that the operator  $\pi e^{rC}$  ( $r > 0$ ) implement a one-to-one mapping of the plane carrying  $P$  onto the plane  $L$ .

**Case II.** Consider  $\max_{u \in P}(\varphi, \pi e^{rC} u)$ ,  $\max_{v \in Q}(\varphi, \pi e^{rC} v)$ ,  $\max_{w \in \hat{w}(r)}(\varphi, w)$ , where  $\varphi$  is an arbitrary unit vector from  $L$ . Suppose that the boundaries of

the sets  $P, Q, \hat{w}(r)$  in their corresponding supporting planes are smooth and strictly convex; then the maxima under consideration are attained at uniquely determined vectors  $u(r, \varphi), v(r, \varphi), w(r, \varphi)$ . From the theory of convex sets follows the equality

$$\pi e^{rC} v(r, \varphi) + w(r, \varphi) = \pi e^{rC} u(r, \varphi). \quad (6)$$

We shall assume the following conditions to be fulfilled:

$$u(r, \varphi) \equiv u(\varphi), \quad v(r, \varphi) \equiv v(\varphi) \quad (0 < r \leq T(z_0)). \quad (7)$$

for any  $\varphi$ . Then from equality (6) it follows that

$$\pi e^{rC} u(\varphi) - \pi e^{rC} v(\varphi) \in \hat{w}(r) \quad (0 < r \leq T(z_0)).$$

The mappings  $u = u(\varphi)$ ,  $v = v(\varphi)$  carry out a homeomorphic mapping of the unit sphere in the plane  $L$  onto the boundaries of the sets  $P$  and  $Q$ , respectively. If  $u \in \partial P$ , then the vector  $\varphi$  for which  $u = u(\varphi)$  is uniquely determined, and from this  $\varphi$  we construct  $v = v(\varphi)$ , which will satisfy condition A. If the point  $u$  lies inside  $P$  (relative to the supporting plane), then we agree to draw through it a straight line with fixed direction vector  $\vec{a}$  belonging to the supporting plane. Then the given point  $u$  is uniquely representable as a linear combination of two boundary points  $u_1(u), u_2(u)$ :  $u = \lambda_1(u)u_1(u) + \lambda_2(u)u_2(u)$ , where  $\lambda_1(u) + \lambda_2(u) = 1$ . Further,  $u_1(u) = u(\varphi_1)$ ,  $u_2(u) = u(\varphi_2)$ . As  $v$ , satisfying condition A, take

$$v = \lambda_1(u)v(\varphi_1) + \lambda_2(u)v(\varphi_2).$$

Let us note that, under the assumptions of problem (2), complete sweeping occurs, and the time  $T(z_0)$  coincides with the time of the upper layer (see (1)). Therefore, if condition (7) is added to the conditions of problem (2), then the time of the upper layer will be optimal.

**7. Examples.** Among the problems satisfying the conditions of case I, the problem of pursuit of objects of the same type is of interest.

Let an object  $x$ , controlled by the system  $\dot{x} = Cx + u$ , where  $x$  is an  $n$ -dimensional vector,  $C$  is a square matrix of order  $n$ ,  $u \in P$ , pursue an object  $y$ , controlled by the system  $\dot{y} = Cy + v$ , where  $y$  is an  $n$ -dimensional vector,  $v \in Q$ . The sets  $P$  and  $Q$  lie in parallel planes. The pursuit is considered completed when  $x(t) - y(t)$  belongs to the linear subspace  $M$ . Making the substitution  $z = x - y$ , we arrive at problem (1).

From the theoretical point of view, case II is more important, since the problems solved with its help are fundamentally more complex. Its conditions are satisfied by the control example of L. S. Pontryagin (see (1)).

Problem (2) satisfies the conditions of both cases.

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## CITED LITERATURE

1. L. S. Pontryagin, *Uspekhi Mat. Nauk*, **21**, no. 4 (130), 219 (1966).
2. E. F. Mishchenko, L. S. Pontryagin, *Dokl. Akad. Nauk SSSR*, **174**, no. 1, 27 (1967).
3. L. S. Pontryagin, *Dokl. Akad. Nauk SSSR*, **174**, no. 6, 1278 (1967).
4. L. S. Pontryagin, *Dokl. Akad. Nauk SSSR*, **175**, no. 4, 764 (1967).
5. B. N. Pshenichnyi, *Avtomatika i Telemekhanika*, no. 1, 65 (1968).

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