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1969

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Abstract

Full Text

UDC 517.947.5.37

MATHEMATICS

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EXPANSION IN EIGENFUNCTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION WITH OPERATOR COEFFICIENTS

(Presented by Academician A. Yu. Ishlinskii, 5 VI 1968)

The results of G. Weyl ⁽¹⁾ are known, concerning expansion in eigenfunctions of the self-adjoint boundary-value problem

$$l[y] = y'' + p(t)y = \lambda y \quad (0 \leq t < \infty), \quad y'(0) - hy(0) = 0,$$

where $p(t)$ and h are real. F. S. Roze-Beketov ⁽²⁾ generalized these results to the case where $p(t)$ and h are bounded self-adjoint operators in a separable Hilbert space H .

In the present paper we consider the case where $p(t)$ is an unbounded operator of the form $p(t) = A - q(t)$, where A is a self-adjoint operator in H , semi-bounded below, and $q(t)$ is a bounded self-adjoint operator. This case includes, for example, eigenvalue problems for certain hyperbolic equations.

1. Let H be a separable Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Denote by $L_2(H, (0, b))$ ($0 < b \leq \infty$) the set of all vector-functions $u(t)$ ($0 \leq t \leq b$) with values in H such that

$$\int_0^b \|u(t)\|^2 dt < \infty.$$

As is known, $L_2(H, (0, b))$ is a complete Hilbert space with scalar product

$$(u, v)_b = \int_0^b (u(t), v(t)) dt \quad (u, v \in L_2(H, (0, b))).$$

Consider the differential equation

$$l[u] = u'' + Au - q(t)u = \lambda u \tag{1}$$

with the boundary condition

$$u'(0) = 0^*, \quad (2)$$

where $q(t) = q^*(t)$ (* denotes passage to the adjoint operator) is an operator-function, continuous in the uniform operator topology, whose values are bounded operators in H ; λ is a complex number; A is a self-adjoint operator in H , semibounded below. Without loss of generality one may assume that $A > 0$ and the operator A^{-1} is bounded. We also assume that the functions $A^{1/2}q(t)A^{-1/2}$ and $Aq(t)A^{-1}$ are strongly continuous in t .

A vector-function $u(t)$ is called a strong solution of equation (1) if $u(t)$, for every t , belongs to $D(A)$ ($D(A)$ is the domain of definition of the operator A), is twice strongly differentiable, and satisfies equation (1).

* The condition (2) is considered for simplicity of exposition. The results are also valid for a boundary condition of the form $u'(0) = Bu(0)$, where B is a bounded self-adjoint operator with the property $BD(A) \subset D(A)$.

On the set $H_+ = D(A)$ introduce the scalar product $(f, g)_+ = (Af, Ag)$. Then H_+ is a complete Hilbert space with respect to $(\cdot, \cdot)_+$, and it may be regarded as a space with positive norm with respect to $H_0 = H$ (see (3)). We denote by H_- the space with negative norm constructed from H_+ and H_0 .

A vector function $u(t)$ with values in H will be called a weak solution of equation (1) if $u(t)$ is twice weakly differentiable in H_- (i.e., the scalar function $(u(t), f)$ is twice differentiable for every $f \in H_+$) and

$$\frac{d^2}{dt^2}(u(t), f) + (u(t), Af) - (u, q(t)f) = \lambda(u(t), f) \quad (f \in H_+).$$

By the method of successive approximations it is not difficult to show that the integral equations

$$\begin{aligned} \omega_1(t, \lambda) &= \cos \sqrt{A - \lambda E} t + \int_0^t \frac{\sin \sqrt{A - \lambda E}(t - x)}{\sqrt{A - \lambda E}} q(x) \omega_1(x, \lambda) dx, \\ \omega_2(t, x, \lambda) &= \frac{\sin \sqrt{A - \lambda E}(t - x)}{\sqrt{A - \lambda E}} + \int_x^t \frac{\sin \sqrt{A - \lambda E}(t - s)}{\sqrt{A - \lambda E}} q(s) \omega_2(s, x, \lambda) ds \end{aligned}$$

(E is the identity operator in H) have solutions in the class of strongly continuous operator functions. For fixed $t, x \in [0, b]$, the solutions $\omega_1(t, \lambda)$ and $\omega_2(t, x, \lambda)$ are entire functions of λ .

If $f \in D(A)$, $g \in D(A^{1/2})$, then $\omega_1(t, \lambda)f$ and $\omega_2(t, 0, \lambda)g$ are strong solutions of equation (1), satisfying the initial data

$$\omega_1(0, \lambda)f = f, \quad \omega_1'(0, \lambda)f = 0; \quad \omega_2(0, 0, \lambda)g = 0, \quad \omega_2'(0, 0, \lambda)g = g.$$

If $f, g \in H$, then $\omega_1(t, \lambda)f$ and $\omega_2(t, 0, \lambda)g$ are weak solutions of this equation.

Theorem 1. The Cauchy problem $u(0) = f$, $u'(0) = g$ ($f, g \in H$) for equation (1) has a unique weak solution. This solution has the form

$$u(t, \lambda) = \omega_1(t, \lambda)f + \omega_2(t, 0, \lambda)g.$$

2. Denote by D' the set of all functions $u(t) \in L_2(H, (0, b))$ that are twice strongly differentiable in H , for each fixed $t \in [0, b]$ belong to $D(A)$, satisfy condition (2), and are such that $l[u] \in L_2(H, (0, b))$. On D' define the operator L' : $L'u = l[u]$. Denote also by D'_0 the set of functions $u(t) \in D'$ finite in a neighborhood of the point b , and by L'_0 the restriction of L' to D'_0 . The operator L'_0 is Hermitian and $L' \subset L'^*_0$. Let L_0 be the closure of L'_0 .

Theorem 2. $D(L_0^*)$ consists of functions $u(t) \in L_2(H, (0, b))$ having two weak derivatives in H_- and such that $u'(0) = 0$. If $L_0^*u = u^*$, then

$$u(t) = \omega_1(t, 0)f + \int_0^t \omega_2(t, x, 0)u^*(x) dx \quad (f = u(0) \in H). \quad (3)$$

In the case $b < \infty$, the operator L_0^* coincides with the closure of the operator L' in $L_2(H, (0, b))$, and $D(L_0)$ consists of functions $u(t) \in L_2(H, (0, b))$, twice weakly differentiable in H_- , and such that $u'(0) = u(b) = u'(b) = 0$.

3. Denote by $\mathfrak{N}_\lambda$ the defect subspace of the operator L_0 . The subspace $\mathfrak{N}_\lambda$ consists of vectors of the form $\omega_1(t, \bar{\lambda})f$, where $f \in H$ is such that

$$\int_0^b \|\omega_1(t, \bar{\lambda})f\|^2 dt < \infty.$$

In what follows we shall assume that $b < \infty$. In this case L_0 has infinite defect numbers.

Put

$$I_\lambda = I_{\lambda, b} = \int_0^b \omega_1^*(t, \lambda)\omega_1(t, \lambda) dt.$$

Lemma 1. The operator I_λ is invertible for every complex λ .

Let $\tilde{L}$ be a self-adjoint extension in $L_2(H, (0, b))$ of the operator L_0 . Then $D(\tilde{L})$ consists of elements of the form

$$y = u + \omega_1(\cdot, \tilde{\lambda}_0) I_{\lambda_0}^{-1/2} U I_{\lambda_0}^{1/2} f - \omega_1(\cdot, \lambda_0) f, \quad (4)$$

where $u \in D(L_0)$; $f \in H$; U is a unitary operator in H ; λ_0 is a fixed nonreal number ($\text{Im } \lambda_0 > 0$). Conversely, if U is a unitary operator in H , then formula (4) defines the domain of a certain self-adjoint extension $\tilde{L}$ of the operator L_0 in $L_2(H, (0, b))$.

Theorem 3. A vector-function $y(t) \in D' \cap D(\tilde{L})$ if and only if it satisfies the boundary condition

$$\begin{aligned} & [\omega_1^*(b, \lambda_0) - I_{\lambda_0}^{1/2} U I_{\tilde{\lambda}_0}^{-1/2} \omega_1^*(b, \tilde{\lambda}_0)] y'(b) - \\ & - [\omega_1'^*(b, \lambda_0) - I_{\lambda_0}^{1/2} U I_{\tilde{\lambda}_0}^{-1/2} \omega_1'^*(b, \tilde{\lambda}_0)] y(b) = 0. \end{aligned} \quad (5)$$

Between the set of all self-adjoint extensions of L_0 in $L_2(H, (0, b))$ and the set of all unitary operators U in H there exists a one-to-one correspondence.

4. We shall say that an operator-function $M(z)$ in H belongs to the class R if it is analytic in the upper half-plane and $\text{Im } M(z) = [M(z) - M^*(z)]/2i \geq 0$ for $\text{Im } z > 0$. An operator-function $M(z)$ of class R is represented uniquely in the form

$$M(z) = P + Qz + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma(t), \quad (6)$$

where P is a bounded self-adjoint operator and Q is a bounded positive operator; $\sigma(t)$ ($\sigma(0) = 0$, $\sigma(t-0) = \sigma(t)$ in the strong sense) is a monotonically increasing operator function such that $\sigma(+\infty)$ is a bounded operator.

Theorem 4. There exists an operator-function $M(z) \in R$ such that

$$\chi(b, \lambda) = \omega_2(b, 0, \lambda) - \omega_1(b, \lambda)M(\lambda) = 0.$$

Denote

$$G(t, s, \lambda) = \begin{cases} \omega_1(t, \lambda)\chi^*(s, \tilde{\lambda}), & \text{for } 0 \leq t \leq s \leq b, \\ \chi(t, \lambda)\omega_1^*(s, \tilde{\lambda}), & \text{for } 0 \leq s \leq t \leq b, \end{cases}$$

where $\chi(t, \lambda) = \omega_2(t, 0, \lambda) - \omega_1(t, \lambda)M(\lambda)$. The function $G(t, s, \lambda)$ has the following properties:

- 1) for fixed s and t , $G(t, s, \lambda)$ is an analytic function of λ ($\text{Im } \lambda \neq 0$);
- 2) for fixed λ , $G(t, s, \lambda)$ ($0 \leq t, s \leq b$) is a strongly continuous function jointly in the variables t and s ;
- 3) for fixed s ($0 \leq s \leq b$), the vector-function $G(t, s, \lambda)f$, $f \in H$, has a weak derivative with respect to t in H in each of the intervals $[0, s)$ and $(s, b]$, and, at $t = s$,

$$\frac{\partial}{\partial t} G(s+0, s, \lambda)f - \frac{\partial}{\partial t} G(s-0, s, \lambda)f = f;$$

- 4) in each of the intervals $[0, s)$ and $(s, b]$, $G(t, s, \lambda)f$, $f \in H$, as a function of t , is a weak solution of equation (1), satisfying the conditions $u'(0) = 0$, $u(b) = 0$.

Let L_1 be the closure in $L_2(H, (0, b))$ of the operator L'_1 , defined on the set of functions $y(t) \in D'$ satisfying the conditions $y'(0) = 0$, $y(b) = 0$, by means of the differential expression $l[y]$. L_1 is a self-adjoint extension of the operator L_0 , and its resolvent, by virtue of properties 1)–4), admits the integral representation

$$R_\lambda y = (L_1 - \lambda E)^{-1} y = \int_0^b G(\cdot, s, \lambda) y(s) ds. \quad (7)$$

5. Introduce the ω_1 -transform of a function $y(t) \in L_2(H, (0, b))$:

$$\tilde{y}(\lambda) = \int_0^b \omega_1^*(t, \bar{\lambda}) y(t) dt.$$

An operator function $\rho(\lambda)$ ($\rho(\lambda - 0) = \rho(\lambda)$, $\rho(0) = 0$) will be called a distribution function if $\rho(\Delta) = \rho(\lambda + \Delta) - \rho(\lambda)$ is a positive bounded operator in H .

Theorem 5. *There exists an operator distribution function $\rho(\lambda)$ ($-\infty < \lambda < \infty$) such that, for any $y(t), z(t) \in L_2(H, (0, b))$,*

$$\int_0^b (y(t), z(t)) dt = \int_{-\infty}^{\infty} (d\rho(\lambda) \tilde{y}(\lambda), \tilde{z}(\lambda)) \quad (8)$$

($d\rho(\lambda) = (1 + \lambda^2) d\sigma(\lambda)$, where $\sigma(\lambda)$ is the function from the representation (6), uniquely determined by $M(z)$)

An operator function $\rho(\lambda)$ for which (8) holds will be called the spectral function of problem (1)–(2) on the interval $[0, b]$.

6. Consider the case when $b = \infty$. If $\beta_1 < \beta_2 < \infty$, then $0 < I_{\lambda, \beta_1} < I_{\lambda, \beta_2}$; on the basis of (3), $I_{\lambda, \beta_1}^{-1} \geq I_{\lambda, \beta_2}^{-1}$, i.e. $I_{\lambda, \beta}^{-1}$ ($0 \leq \beta < \infty$) is a monotonically decreasing operator function in H . Therefore, as $\beta \rightarrow \infty$, it converges to a positive bounded operator Γ_λ . Similarly to how this is done in ⁴, one can prove the following theorems.

Theorem 6. *For each λ ($\text{Im } \lambda \neq 0$) there exists an operator solution $x(t, \lambda)$ of equation (1) such that, for all $f \in H$,*

$$\int_0^\infty \|x(t, \lambda)f\|^2 dt < \infty.$$

Theorem 7. *The dimension of the orthogonal complement to the null manifold of the operator Γ_λ ($\text{Im } \lambda \neq 0$) coincides with the dimension of the defect subspace $\mathfrak{N}_\lambda$ of the operator L_0 , and hence is the same for all λ from a connected component of the regularity field of the operator L_0 .*

Using Theorem 5 for arbitrary $\beta < \infty$, by the method of stretching the interval (see ⁵) we arrive at the assertion.

Theorem 8. *There exists a spectral function $\rho(\lambda)$ of problem (1)–(2) on the half-axis $(0, \infty)$, i.e. one such that, for any $y, z \in L_2(H, (0, b))$,*

$$\int_0^\infty (y(t), z(t)) dt = \int_{-\infty}^\infty (d\rho(\lambda) \tilde{y}(\lambda), \tilde{z}(\lambda)) \left(\int_{-\infty}^\infty \frac{(d\rho(\lambda)f, f)}{1 + \lambda^2} < \infty \right).$$

In conclusion the authors express their gratitude to Yu. M. Berezanskii for his attention to this work.

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Received
30 V 1968

REFERENCES

1. H. Weyl, Math. Ann., 68, 222 (1910).
2. F. S. Roze-Beketov, Matem. sborn., 51 (93), No. 3, 293 (1960).
3. Yu. M. Berezanskii, *Expansion in Eigenfunctions of Self-Adjoint Operators*, Kiev, 1965.
4. M. L. Gorbachuk, Ukr. matem. zhurn., 18, No. 2, 3 (1966).
5. B. M. Levitan, *Expansion in Eigenfunctions of Second-Order Differential Equations*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.

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