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S. L. Krushkal'

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Abstract

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MATHEMATICS

S. L. KRUSHKAL

AN APPROXIMATION THEOREM FOR ANALYTIC FUNCTIONS AND ITS APPLICATIONS

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In this note an approximation theorem for analytic functions is proved and its applications in the theory of quasiconformal mappings are indicated. Its proof is based on certain considerations used in ^(1,3).

1. For a nonnegative integer q , denote by $A_q(U)$ the complex Banach space of functions $\varphi(z)$ analytic in the disk $U : |z| < 1$, with norm

$$\|\varphi\|_{A_q(U)} = \iint_U (1 - |z|^2)^q |\varphi(z)| dx dy < \infty.$$

Let n be a fixed natural number.

Theorem 1. For any function $\varphi(z) \in A_q(U)$ there exists a sequence of rational functions $r_j(z)$, having, possibly, simple poles, and only at points of the circumference $\gamma : |z| = 1$, satisfying the condition $\text{Im}[z^n r_j(z)] = 0$ on γ (when $r_j(z) \neq \infty$) and such that

$$\lim_{j \rightarrow \infty} \iint_U (1 - |z|^2)^q |r_j(z) - \varphi(z)| dx dy = 0. \quad (1)$$

Proof. Consider the space $A_q(U)$ as a real Banach space with the same norm (allowing multiplication of functions $\varphi(z) \in A_q(U)$ only by real numbers) and denote this real space by $\tilde{A}_q(U)$.

Let α be the set of all rational functions $r(z)$ with simple poles lying on γ , satisfying the condition $\text{Im}[z^n r(z)] = 0$ for $|z| = 1$ (if $r(z) \neq \infty$). The set α is linear in $\tilde{A}_q(U)$. To prove the theorem it suffices to show that if l is any linear (real) functional in $\tilde{A}_q(U)$ such that $l(\varphi) = 0$ for all $\varphi \in \alpha$, then $l = 0$ in $\tilde{A}_q(U)$.

For the given functional $l(\varphi)$ in $\tilde{A}_q(U)$, put $L(\varphi) = l(\varphi) - il(i\varphi)$. Then $L(\varphi)$ is a complex linear (bounded) functional in $A_q(U)$, and $l(\varphi) = \text{Re } L(\varphi)$. By the

Hahn-Banach and F. Riesz-Steinhaus theorems, the functional $L(\varphi)$ has the form

$$L(\varphi) = \iint_U (1 - |z|^2)^q \nu(z) \varphi(z) dx dy = \iint_U \mu(z) \varphi(z) dx dy, \quad (2)$$

where $\mu(z)$ is some complex-valued function measurable in U such that $(1 - |z|^2)^{-q} \mu(z) = \nu(z) \in L_\infty(U)$. Put

$$h(z) = -\frac{z^{2n-1} - 1}{\pi} \iint_{|\zeta| < 1} \frac{\mu(\zeta) d\xi d\eta}{(\zeta - z)(\zeta^{2n-1} - 1)}, \quad h_1(z) = (1 + z)h(z), \quad (3)$$

where $\zeta = \xi + i\eta$. Let $a_k = e^{2k\pi i/(2n-1)}$, $k = 0, 1, \dots, 2n-2$. On the basis of the known properties of the integral

$$-\frac{1}{\pi} \iint_U \frac{\mu(\zeta) d\xi d\eta}{\zeta - z}$$

for $\mu \in L_\infty(U)$ (see (4), Ch. I, § 6), we conclude that the function $h(z)$ has the following properties:

- 1) $h(a_k) = 0$, $k = 0, 1, \dots, 2n-2$;
- 2) $h(z)$ is continuous in the entire z -plane, and for every disk $U_R : |z| < R$, $R > 0$, we have

$$|h(z_1) - h(z_2)| \leq C(R) |z_1 - z_2| |\ln |z_1 - z_2||, \quad |z_1|, |z_2| \leq R, \quad C(R) = \text{const}; \quad (4)$$

- 3) $h(z)$ has generalized derivatives in the sense of S. L. Sobolev, $h_z, h_{\bar{z}} \in L_p(U_R)$, $p > 1$, with $h_{\bar{z}} = \mu$ for $z \in U$;
- 4) $h(z)$ is analytic for $|z| > 1$, and $h(z) = O(|z|^{2n-2})$ near $z = \infty$.

Let $l(\varphi) = 0$ for all $\varphi \in \alpha$. For any fixed $z \in \gamma$, $z \neq a_k$ ($k = 0, 1, \dots, 2n-2$), the function $\Phi_{(z)}(\xi) = (1 + z)(\xi - z)^{-1}(\xi^{2n-1} - 1)^{-1}$ belongs to the set α , and

$$-\pi h_1(z) = L(\Phi_{(z)})(z^{2n-1} - 1).$$

Put $h_2(z) = -\pi h_1(z)/(z^{2n-1} - 1)$. The function $h_2(z)$ at the points $a_0, a_1, \dots, a_{2n-2}$ may tend to infinity logarithmically; it is continuous at the remaining points of the circle $|z| = 1$ and bounded at infinity. Since $l(\Phi_{(z)}) = 0$, we have $\text{Re } h_2(z) = 0$ for $|z| = 1$, $z \neq a_k$, $k = 0, 1, \dots, 2n-2$. Hence, by the Poisson formula for harmonic functions, we obtain that $\text{Re } h_2(z) = 0$ everywhere for $|z| \geq 1$. Consequently, for $|z| \geq 1$ we have $h_2(z) = iC$, where C is a real constant. But $h_2(-1) = 0$, and therefore $h_2(z) = 0$ for $|z| \geq 1$. It follows that

$$h_1(z) = h(z) = 0 \quad \text{for } |z| \geq 1. \quad (5)$$

Denote by $\delta(z) = 1 - |z|$ the distance from the point $z \in U$ to γ , assuming $\delta \leq \delta_0 \leq e^{-2}$. From (4) and (5) it follows that

$$|h(z)| \leq C_1 \delta(z) \ln \frac{1}{\delta(z)},$$

where $C_1 = C(1)$, $|z| < 1$.

We shall prove that for $q \geq 1$ the stronger estimate

$$|h(z)| \leq 2C_1 [\delta(z)]^{q+1} \ln \frac{1}{\delta(z)}, \quad |z| < 1 \quad (6)$$

holds.

For this purpose, for an arbitrary fixed θ , $0 \leq \theta \leq 2\pi$, consider the function

$$\tilde{h}(z) = -\frac{(1 - ze^{i\theta})^q (z^{2n-1} - 1)}{\pi} \iint \frac{(1 - |\zeta|^2)^q \nu(\zeta) d\xi d\eta}{(1 - \zeta e^{i\theta})^q (\zeta - z)(\zeta^{2n-1} - 1)}. \quad (7)$$

Since the generalized derivative $h_{\bar{z}} - \tilde{h}_{\bar{z}} = 0$ in any disk $|z| \leq R$, $R > 0$, we have $h(z) = \tilde{h}(z) + \Phi(z)$, where $\Phi(z)$ is a polynomial of degree $q + 2n - 2$. By virtue of (5), $\tilde{h}(z) = -\Phi(z)$ for $|z| \geq 1$, and from (7) it is clear that $\tilde{h}(z)$ has zeros $z = e^{-i\theta}$ and $z = a_k$, $k = 0, 1, \dots, 2n - 2$, of total multiplicity $q + 2n - 1$. Consequently, $\Phi(z) \equiv 0$, i.e. $h(z) \equiv \tilde{h}(z)$. Hence, applying to the integral on the right-hand side of (7) an inequality analogous to (4), and putting $\theta = -\arg z$, we obtain (6).

Following L. Ahlfors ⁽¹⁾, we now consider an infinitely differentiable function $\Lambda(t)$ such that $\Lambda(t) = 0$ for $0 \leq t \leq 1$, $\Lambda(t) = 1$ for $t \geq 2$, and for an arbitrary positive $\varepsilon > 0$ put

$$\lambda_\varepsilon(z) = \Lambda[\varepsilon^{-1}(\ln \ln 1/\delta(z))^{-1}]. \quad (8)$$

Taking into account that $|\partial\delta(z)/\partial\bar{z}| \leq 1$, $\Lambda'(t) = 0$ outside the interval $1 \leq t \leq 2$, and $(\ln \ln 1/\delta)^{-1} = t \leq 2\varepsilon$ for $t \in [\varepsilon, 2\varepsilon]$, we shall have

$$|\partial\lambda_\varepsilon/\partial\bar{z}| \leq C_2 \varepsilon^{-1} \delta^{-1} (\ln 1/\delta)^{-1} (\ln \ln 1/\delta)^{-2} \leq 4C_2 \varepsilon \delta^{-1} (\ln 1/\delta)^{-1}, \quad (9)$$

where $C_2 = \max |\Lambda'(t)|$ for $1 \leq t \leq 2$. From (6) and (9) we have

$$|\lambda_{\varepsilon\bar{z}} h(z)| \leq 8C_1 C_2 \varepsilon (1 - |z|^2)^q, \quad |z| < 1. \quad (10)$$

By Green's formula, in view of the finiteness of $\lambda_\varepsilon(z)$, for any function $\varphi \in A_q(U)$ we obtain

$$\iint_U \lambda_\varepsilon \mu \varphi dx dy = \iint_U \lambda_\varepsilon (h\varphi)_{\bar{z}} dx dy - \iint_U \lambda_{\varepsilon\bar{z}} h\varphi dx dy$$

and, consequently, by virtue of (10),

$$\left| \iint_U \lambda_\varepsilon(z) \mu(z) \varphi(z) dx dy \right| \leq C_3 \varepsilon \|\varphi\|_{A_q(U)}, \quad C_3 = 8C_1 C_2. \quad (11)$$

From (11), in view of the equality $\lim \lambda_\varepsilon(z) = 1$ in the limit as $\varepsilon \rightarrow 0$, taking (2) into account, we obtain that $L(\varphi) = 0$, whence it follows that also $l(\varphi) = 0$. Thus the theorem is proved.

Remark. As is seen from the proof, the assertion of Theorem 1 remains valid also under the condition that the functions $r_j(z)$ may have poles only at the points of some set $E \subset \gamma$, everywhere dense on γ and containing the points $a_0, a_1, \dots, a_{2n-2}$.

2. We indicate some applications of Theorem 1. Since for $|z| = 1$ we have $z = e^{i\psi}$ and $\operatorname{Im}[r(z) dz^2] = -\operatorname{Im}[z^2 r(z)] d\psi^2$, it follows from Theorem 1 that

Theorem 2. *For any function $\varphi(z) \in A_q(U)$ there exists a sequence of rational functions $r_j(z)$, which may have simple poles, and moreover only at points of the circle $\gamma: |z| = 1$, satisfying the condition $\operatorname{Im}[r_j(z) dz^2] = 0$ on γ , and such that*

$$\lim_{j \rightarrow \infty} \|r_j(z) - \varphi(z)\|_{A_q(U)} = 0.$$

Theorem 2 makes it possible to establish some properties of extremal quasiconformal mappings.

For fixed q , denote by Ω_q the set of all homeomorphisms $w = \omega(z)$ of the circle γ onto $\gamma': |w| = 1$ that are boundary values of quasiconformal homeomorphisms $w = f^\mu(z)$ of the disk U onto $U': |w| < 1$ with complex characteristics $\mu(z) = k\varphi(z)/|\varphi(z)|$, where $\varphi \in A_q(U)$, and $k = \|\varphi\|_{A_q(U)} < 1$. Without loss of generality, we shall assume that $\omega(e^{2\pi li/3}) = e^{2\pi li/3}$, $l = 0, 1, 2$. Denote by $w = f^{\mu_m}(z)$ the mapping (with characteristic $\mu_m(z)$) minimizing the dilatation $K(f)$ in the class of all quasiconformal homeomorphisms of the disk U onto U' , sending the given (distinct) points $z_1, z_2, \dots, z_m$ of the circle γ to the points $\omega(z_1), \omega(z_2), \dots, \omega(z_m)$, respectively, for a given $\omega \in \Omega_q$, and call such mappings extremal.* Then the characteristic $\mu_m(z) = k_m r_m(z)/|r_m(z)|$, where $k_m = \operatorname{const} < 1$, and $r_m(z)$ is a rational function which may have simple poles at the points $z_1, z_2, \dots, z_m$ and is such that $\operatorname{Im}[r_m(z) dz^2] = 0$ for $|z| = 1$, while $r_m(z)$ is determined up to a constant positive factor (see $(^2, ^5, ^7)$). We choose this factor so that $\|r_m\|_{A_q(U)} = k_m$. Conversely, by virtue of what was proved in $(^2, ^7)$, every mapping $w = f^{\mu_m}(z)$ with characteristic $\mu_m(z) = \|r_m\|_{A_q(U)} r_m(z)/|r_m(z)|$ of the indicated form is extremal in the class of quasiconformal homeomorphisms of the disk U onto U' sending the points $z_1, z_2, \dots, z_m$ of the circle γ , which are poles of $r_m(z)$, to the points $f^{\mu_m}(z_1), f^{\mu_m}(z_2), \dots, f^{\mu_m}(z_m)$, respectively.

From Theorem 1 the assertion follows.

Every mapping $w = f^\mu(z)$ with characteristic $\mu = \bar{k}\varphi/|\varphi|$, $k < 1$, with $\varphi \in A_q(U)$, is the limit, in the sense of uniform convergence in $\bar{U}$, of some sequence of extremal mappings $w = f^{\mu_m}(z)$ with characteristics $\mu_m(z) = k_m \overline{r_m(z)}/|r_m(z)|$ of the above-indicated form, where $r_m \rightarrow \varphi$ in $A_q(U)$.

* The dilatation of a mapping f with characteristic $\nu(z) = f_{\bar{z}}/f_z$, where $\|\nu\|_{L_\infty(U)} = k(f) < 1$, is the quantity

Here the question remains open whether, for $q \geq 1$, every mapping $w = f^\mu(z)$ with characteristic $\mu = k\varphi/|\varphi|$, $\varphi \in A_q(U)$, is extremal (i.e., minimizes the dilatation $K(f)$) in the class of all quasiconformal mappings f of the disk U onto U' that coincide with f^μ on γ . (This has been proved only for $q = 0$ ⁽⁶⁾.)

Institute of Mathematics
Siberian Branch
Academy of Sciences of the USSR

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References Cited

- ¹ L. Ahlfors, *Am. J. Math.*, **86**, No. 2, 413 (1964).
- ² L. Ahlfors, L. Bers, *Spaces of Riemann Surfaces and Quasiconformal Mappings*, II, 1961, pp. 9 and 104.
- ³ L. Bers, *J. Anal. Math.*, **14**, 1 (1965).
- ⁴ I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.
- ⁵ S. L. Krushkal', *Siberian Mathematical Journal*, **8**, No. 2, 313 (1967).
- ⁶ K. Strebel, *Comm. Math. Helv.*, **39**, Nos. 2-3, 81 (1964).
- ⁷ O. Teichmüller, *Abh. Preuss. Akad. Wiss., Math.-Naturwiss. Kl.*, **22**, 1 (1940).

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